

MATCHINGS IN  $k$ -PARTITE  $k$ -UNIFORM HYPERGRAPHS

JIE HAN, CHUANYUN ZANG, AND YI ZHAO

**ABSTRACT.** For  $k \geq 3$  and  $\epsilon > 0$ , let  $H$  be a  $k$ -partite  $k$ -graph with parts  $V_1, \dots, V_k$  each of size  $n$ , where  $n$  is sufficiently large. Assume that for each  $i \in [k]$ , every  $(k-1)$ -set in  $\prod_{j \in [k] \setminus \{i\}} V_j$  lies in at least  $a_i$  edges, and  $a_1 \geq a_2 \geq \dots \geq a_k$ . We show that if  $a_1, a_2 \geq \epsilon n$ , then  $H$  contains a matching of size  $\min\{n-1, \sum_{i \in [k]} a_i\}$ . In particular,  $H$  contains a matching of size  $n-1$  if each crossing  $(k-1)$ -set lies in at least  $\lceil n/k \rceil$  edges, or each crossing  $(k-1)$ -set lies in at least  $\lfloor n/k \rfloor$  edges and  $n \equiv 1 \pmod k$ . The former case answers a question of Rödl and Ruciński and was independently obtained by Lu, Wang, and Yu. Our proof is more involved than the one used by the first author, who proved the non-partite version of the result, and the one by Lu, Wang, and Yu. In particular, we used a more complex absorbing method and deal with two extremal cases separately.

## 1. INTRODUCTION

A  $k$ -uniform hypergraph (in short,  $k$ -graph) consists of a vertex set  $V$  and an edge set  $E \subseteq \binom{V}{k}$ , that is, every edge is a  $k$ -element subset of  $V$ . A  $k$ -graph  $H$  is  $k$ -partite if  $V(H)$  can be partitioned into  $k$  parts,  $V(H) = V_1 \cup \dots \cup V_k$  such that every edge consists of exactly one vertex from each class, that is,  $E(H) \subset V_1 \times \dots \times V_k$ . A *matching* in  $H$  is a collection of vertex-disjoint edges of  $H$ . A matching covering all vertices of  $H$  is called *perfect*.

Given a  $k$ -graph  $H$  and a set  $S$  of  $d$  vertices in  $V(H)$ ,  $1 \leq d \leq k-1$ , a *neighbor* of  $S$  is a  $(k-d)$ -set  $T \subseteq V(H) \setminus S$  such that  $S \cup T \in E(H)$ . Denote by  $N_H(S)$  the set of the neighbors of  $S$ , and define the *degree* of  $S$  to be  $\deg_H(S) = |N_H(S)|$ . We omit the subscript  $H$  if it is clear from the context. The *minimum  $d$ -degree*  $\delta_d(H)$  of  $H$  is the minimum of  $\deg_H(S)$  over all  $d$ -subsets  $S$  of  $V(H)$ .

The minimum  $d$ -degree thresholds that force a perfect matching in  $k$ -graphs have been studied intensively, see [2, 3, 5, 9, 12, 13, 15, 18, 19, 22, 23, 24, 25, 26] and surveys [20, 28]. In particular, Rödl, Ruciński and Szemerédi [23] determined the minimum codegree threshold that guarantees a perfect matching in an  $n$ -vertex  $k$ -graph for large  $n$  and all  $k \geq 3$ . The threshold is  $n/2 - k + C$ , where  $C \in \{3/2, 2, 5/2, 3\}$  depends on the values of  $n$  and  $k$ . In contrast, when  $n$  is not divisible by  $k$ , they proved that a  $k$ -graph  $H$  on  $n$  vertices satisfying  $\delta_{k-1}(H) \geq n/k + O(\log n)$  contains a matching of size  $\lfloor n/k \rfloor$ . Recently the first author [7] improved this result by showing that  $\delta_{k-1}(H) \geq \lfloor n/k \rfloor$  suffices.

In this paper we are interested in the corresponding thresholds in  $k$ -partite  $k$ -graphs. Suppose  $H$  is a  $k$ -partite  $k$ -graph with  $V(H) = V_1 \cup \dots \cup V_k$ . A subset  $S \subset V(H)$  is called *crossing* if  $|S \cap V_i| \leq 1$  for each  $i \in [k]$ . For any  $I \subseteq [k]$ , let  $\delta_I(H)$  be the minimum of  $\deg_H(S)$  taken over all crossing  $|I|$ -vertex sets  $S$  in  $\prod_{i \in I} V_i$ . Then the *partite minimum  $d$ -degree*  $\delta'_d(H)$  is defined as the minimum of  $\delta_I(H)$  over all  $d$ -element sets  $I \subseteq [k]$ .

Let  $H$  be a  $k$ -partite  $k$ -graph with  $n$  vertices in each part. For  $k \geq 3$ , Kühn and Osthus [14] proved that if  $\delta'_{k-1}(H) \geq n/2 + \sqrt{2n \log n}$  then  $H$  has a perfect matching. Later Aharoni, Geogakopoulos and Sprüsel [1] improved this result by requiring only two partite minimum codegrees. They showed that  $H$  contains a perfect matching if  $\delta_{[k] \setminus \{1\}}(H) > n/2$  and  $\delta_{[k] \setminus \{2\}}(H) \geq n/2$ , and consequently, if  $\delta'_{k-1}(H) > n/2$  then  $H$  has a perfect matching.

Similar to the non-partite case, when targeting on almost perfect matchings, the minimum degree threshold also drops significantly. Kühn and Osthus in [14] proved that  $\delta'_{k-1}(H) \geq \lceil n/k \rceil$  guarantees a matching of size  $n - (k-2)$ . Rödl and Ruciński [20, Problem 3.14] asked whether  $\delta'_{k-1}(H) \geq \lceil n/k \rceil$  guarantees a matching

*Date:* November 2, 2016.

*2010 Mathematics Subject Classification.* Primary 05C70, 05C65.

*Key words and phrases.* matching, hypergraph, absorbing method.

The first author is supported by FAPESP (Proc. 2013/03447-6, 2014/18641-5, 2015/07869-8). The third author is partially supported by NSF grant DMS-1400073.

in  $H$  of size  $n - 1$ . In this paper, we answer this question in the affirmative and show that the threshold can be actually weakened to  $\lfloor n/k \rfloor$  if  $n \equiv 1 \pmod{k}$ . In fact, our result is much more general – it only requires that the sum of the partite minimum codegrees is large and at least two partite codegrees are not small.

**Theorem 1.1** (Main Result). *For any  $k \geq 3$  and  $\epsilon > 0$ , there exists  $n_0$  such that the following holds for all  $n \geq n_0$ . Let  $H$  be a  $k$ -partite  $k$ -graph with parts of size  $n$  and  $a_i := \delta_{[k] \setminus \{i\}}(H)$  for all  $i \in [k]$  such that  $a_1 \geq a_2 \geq \dots \geq a_k$  and  $a_2 > \epsilon n$ . Then  $H$  contains a matching of size at least  $\min\{n - 1, \sum_{i \in [k]} a_i\}$ .*

Our proof, based on the absorbing method, unfortunately fails when  $a_1$  is huge and all of  $a_2, \dots, a_k$  are small. It is unclear (to us) if the same assertion holds in this case.

The following corollary follows from Theorem 1.1 immediately. It appears in the dissertation of the second author [27]. The second case of Corollary 1.2 resolves [20, Problem 3.14] and was independently proven by Lu, Wang and Yu [17].

**Corollary 1.2.** *Given  $k \geq 3$ , there exists  $n_0$  such that the following holds for all  $n \geq n_0$ . Let  $H$  be a  $k$ -partite  $k$ -graph with parts of size  $n$ . Then  $H$  contains a matching of size  $n - 1$  if one of the following holds.*

- $n \equiv 1 \pmod{k}$  and  $\delta'_{k-1}(H) \geq \lfloor n/k \rfloor$ ;
- $\delta'_{k-1}(H) \geq \lceil n/k \rceil$ .

Let  $\nu(H)$  be the size of a maximum matching in  $H$ . The following greedy algorithm, which essentially comes from [14], gives a simple proof of Theorem 1.1 when  $\sum_{i \in [k]} a_i \leq n - k + 2$  or when  $a_1 + a_2 \geq n - 1$ .

**Fact 1.3.** *Let  $n \geq k - 2$ . Suppose  $H$  is a  $k$ -partite  $k$ -graph with parts of size at least  $n$ . Let  $a_i := \delta_{[k] \setminus \{i\}}(H)$  for  $i \in [k]$ . Then*

$$\nu(H) \geq \min \left\{ n - k + 2, \sum_{i \in [k]} a_i \right\} \quad \text{and} \quad \nu(H) \geq \min\{n - 1, a_1 + a_2\}.$$

*Proof.* Assume a maximum matching  $M$  of  $H$  has size  $|M| \leq \min\{n - k + 1, \sum_{i \in [k]} a_i - 1\}$ . Since each class has at least  $k - 1$  vertices unmatched, we can find  $k$  disjoint crossing  $(k - 1)$ -sets  $U_1, \dots, U_k$  such that  $U_i$  contains exactly one unmatched vertex in  $V_j$  for  $j \neq i$ . Each  $U_i$  has at least  $a_i$  neighbors and all of them lie entirely in  $V(M)$ . Since  $\sum_{i \in [k]} a_i > |M|$ , there exist distinct indices  $i \neq j$  such that  $U_i$  and  $U_j$  have neighbors on the same edge  $e \in M$ , say  $v_i \in N(U_i) \cap e$  and  $v_j \in N(U_j) \cap e$ . Replacing  $e$  by  $\{v_i\} \cup U_i$  and  $\{v_j\} \cup U_j$  gives a larger matching, a contradiction. The second inequality can be proved similarly.  $\square$

The following construction, sometimes called a *space barrier*, shows that the degree sum conditions in Theorem 1.1 and Fact 1.3 are best possible. Let  $H_0$  be a  $k$ -partite  $k$ -graph with  $n$  vertices in each part  $V_1, \dots, V_k$ . For  $i \in [k]$ , let  $A_i \subseteq V_i$  be a subset of size  $a_i$  and a crossing  $k$ -set  $e \in E(H_0)$  if and only if  $e \cap A_i \neq \emptyset$  for some  $i \in [k]$ . Suppose  $\sum_{i=1}^k a_i \leq n - 1$ . Clearly both  $\nu(H_0)$  and the partite degree sum of  $H_0$  equal to  $\sum_{i \in [k]} a_i$  (so we cannot expect a matching larger than  $\sum_{i \in [k]} a_i$ ).

To see that we cannot expect for a perfect matching when  $\sum_{i \in [k]} a_i \geq n$ , consider the following example, sometimes called a *divisibility barrier*. Let  $H_1$  be a  $k$ -partite  $k$ -graph with  $n$  vertices in each of its parts  $V_1, \dots, V_k$ . For  $i \in [k]$ , let  $V_i = A_i \cup B_i$  such that  $\sum_{i \in [k]} |A_i|$  is odd and for each  $i \in [k]$ ,  $n/2 - 1 \leq |A_i| \leq n/2 + 1$ . Let  $E(H_1)$  be the set of crossing  $k$ -sets that contain an even number of vertices in  $\bigcup_{i \in [k]} A_i$ . So the partite degree sum of  $H_1$  is at least  $k(n/2 - 1)$ . However,  $H_1$  does not contain a perfect matching because any matching in  $H_1$  covers even number of vertices in  $\bigcup_{i \in [k]} A_i$  but  $|\bigcup_{i \in [k]} A_i|$  is odd.

When proving Corollary 1.2 directly, the authors of [17, 27] closely followed the approach used by the first author [7] by separating the case when  $H$  is close to  $H_0$  from the remaining case. In contrast, to prove Theorem 1.1, we have to consider *three* cases separately: when  $H$  is close to  $H_0$ , when  $H$  is close to (a weaker form of)  $H_1$ , and when  $H$  is far from both  $H_0$  and  $H_1$ .

Given a set  $V$ , let  $V_1 \cup \dots \cup V_k$  and  $A \cup B$  be two partitions of  $V$ . For  $i \in [k]$  we always write  $A_i := A \cap V_i$  and  $B_i := B \cap V_i$ . A set  $S \subseteq V$  is *even* (otherwise *odd*) if it intersects  $A$  in an even number of vertices. Let  $E_{\text{even}}(A, B)$  (respectively,  $E_{\text{odd}}(A, B)$ ) denote the family of all crossing  $k$ -subsets of  $V$  that are even (respectively, odd).

Now we define two extremal cases formally. Let  $H$  be a  $k$ -partite  $k$ -graph with each part of size  $n$  and let  $a_i := \delta_{[k] \setminus \{i\}}(H)$  for all  $i \in [k]$ . Then  $H$  is called  $\epsilon$ -*S-extremal* if  $V(H)$  contains an independent set  $I$  such

that  $|I \cap V_i| \geq n - a_i - \epsilon n$  for each  $i \in [k]$ . On the other hand,  $H$  is called  $\epsilon$ - $D$ -extremal if there is a partition  $A \cup B$  of  $V(H)$  such that

- (i)  $(1/2 - \epsilon)n \leq a_1, a_2, |A_1|, |A_2| \leq (1/2 + \epsilon)n$ ,
- (ii)  $|E_{\text{even}}(A, B) \setminus E| \leq \epsilon n^k$  or  $|E_{\text{odd}}(A, B) \setminus E| \leq \epsilon n^k$ .

**Theorem 1.4** (Non-extremal case). *For an integer  $k \geq 3$  and  $0 < \gamma \ll \epsilon \ll 1/k$ , the following holds for sufficiently large  $n$ . Let  $H$  be a  $k$ -partite  $k$ -graph with parts of size  $n$  with  $a_i := \delta_{[k] \setminus \{i\}}(H)$  for  $i \in [k]$  such that  $(1 - \epsilon)n \geq a_1 \geq a_2 \geq \dots \geq a_k$ ,  $a_2 \geq \epsilon n$ , and  $\sum_{i \in [k]} a_i \geq (1 - \gamma/5)n$ . Then one of the following holds:*

- (i)  $H$  contains a matching of size at least  $n - 1$ ;
- (ii)  $H$  is  $\gamma$ - $S$ -extremal;
- (iii)  $H$  is  $2k^2\epsilon$ - $D$ -extremal.

**Theorem 1.5** (Extremal case I). *For an integer  $k \geq 3$  and  $0 < \gamma \ll \epsilon \ll 1/k$ , the following holds for sufficiently large  $n$ . Let  $H$  be a  $k$ -partite  $k$ -graph with parts of size  $n$  and  $a_i := \delta_{[k] \setminus \{i\}}(H)$  for  $i \in [k]$  such that  $(1 - \epsilon)n \geq a_1 \geq a_2 \geq \dots \geq a_k$  and  $a_2 \geq \epsilon n$ . Suppose  $H$  is  $\gamma$ - $S$ -extremal. Then one of the following holds:*

- (i)  $H$  contains a matching of size at least  $\min\{n - 1, \sum_{i \in [k]} a_i\}$ ;
- (ii)  $H$  is  $3\epsilon$ - $D$ -extremal.

**Theorem 1.6** (Extremal case II). *For any integer  $k \geq 3$  and  $0 < \epsilon \ll 1/k$ , the following holds for sufficiently large  $n$ . Suppose  $H$  is a  $k$ -partite  $k$ -graph with parts of size  $n$  and  $a_i := \delta_{[k] \setminus \{i\}}(H)$  for  $i \in [k]$ . If  $H$  is  $\epsilon$ - $D$ -extremal, then  $H$  contains a matching of size  $\min\{n - 1, \sum_{i \in [k]} a_i\}$ .*

*Proof of Theorem 1.1.* When  $\sum_{i \in [k]} a_i \leq n - k + 2$  or  $a_1 \geq (1 - \epsilon)n$ , Theorem 1.1 follows from Fact 1.3 immediately. When  $\sum_{i \in [k]} a_i \geq n - k + 3$  and  $a_1 \leq (1 - \epsilon)n$ , it follows from Theorems 1.4 – 1.6.  $\square$

The rest of the paper is organized as follows. In Section 2 we introduce two absorbing lemmas that are needed for the proof of Theorem 1.4: one is a simple  $k$ -partite version of [23, Fact 2.3]; the other one is derived from a more involved approach by considering the lattice generated by the edges of  $H$ . In Sections 3 – 5, we give the proofs of Theorems 1.4 – 1.6, respectively.

**Notation:** For any integer  $k \geq 1$ , we write  $[k] := \{1, \dots, k\}$  and for integers  $k \leq k'$  we write  $[k, k'] := \{k, k + 1, \dots, k'\}$ . Throughout this paper, we denote by  $H$  a  $k$ -partite  $k$ -graph with the vertex partition  $V(H) = V_1 \cup \dots \cup V_k$ . A vertex set  $S$  is called *balanced* if it consists of an equal number of vertices from each part of  $V(H)$ . For a  $k$ -graph  $H$  and a set  $W \subseteq V(H)$ , let  $H \setminus W$  be the subgraph of  $H$  induced on  $V(H) \setminus W$ . Throughout the paper, we write  $0 < \alpha \ll \beta \ll \gamma$  to mean that we can choose the constants  $\alpha, \beta, \gamma$  from right to left. More precisely, there are increasing functions  $f$  and  $g$  such that, given  $\gamma$ , whenever  $\beta \leq f(\gamma)$  and  $\alpha \leq g(\beta)$ , the subsequent statement holds. Hierarchies of other lengths are defined similarly.

## 2. ABSORBING TECHNIQUES IN $k$ -PARTITE $k$ -GRAPHS

The main tool in the proof of Theorem 1.4 is the absorbing method. This technique was initiated by Rödl, Ruciński and Szemerédi [21] and has proven to be a powerful tool for finding spanning structures in graphs and hypergraphs. In this section, we prove the absorbing lemmas that will be used in the proof of Theorem 1.4. In fact, we present two different notions of absorbing sets and use them in two different cases.

Let  $H$  be a  $k$ -partite  $k$ -graph. Given a balanced  $2k$ -set  $S$ , an edge  $e \in E(H)$  disjoint from  $S$  is called  *$S$ -absorbing* if there are two disjoint edges  $e_1, e_2 \subseteq S \cup \{e\}$  such that  $|e_1 \cap S| = k - 1$ ,  $|e_1 \cap e| = 1$ ,  $|e_2 \cap S| = 2$ , and  $|e_2 \cap e| = k - 2$ . Given a crossing  $k$ -set  $S$ , a set  $T \subset V(H) \setminus S$  is called  *$S$ -perfect-absorbing* if  $T$  is balanced and both  $H[T]$  and  $H[S \cup T]$  contain perfect matchings.

We first prove the following proposition, which is a standard application of Chernoff's bound.

**Proposition 2.1.** *Given  $\lambda > 0$  and  $i_0 \in \mathbb{N}$ , the following holds for sufficiently large  $n$ . Let  $V$  be a vertex set with  $k$  parts each of size  $n$ , and let  $\mathcal{F}_1, \dots, \mathcal{F}_t$  be families of balanced  $i_0 k$ -sets on  $V$  each of size at least  $\lambda n^{i_0 k}$  and  $t \leq n^{2k}$ . Then there exists a family  $\mathcal{F}'$  of disjoint balanced  $i_0 k$ -sets on  $V$  of size at most  $\lambda n / (4i_0 k)$  such that  $|\mathcal{F}_i \cap \mathcal{F}'| \geq \lambda^2 n / (32i_0 k)$  for each  $i \in [t]$ .*

*Proof.* We build  $\mathcal{F}'$  by standard probabilistic arguments. Choose a collection  $\mathcal{F}$  of balanced  $i_0k$ -sets in  $H$  by selecting each balanced  $i_0k$ -set on  $V$  independently and randomly with probability  $p = \epsilon/(2n^{i_0k-1})$ , where  $\epsilon = \lambda/(4i_0k)$ . Since  $t \leq n^{2k}$ , Chernoff's bound implies that, with probability  $1 - o(1)$ , the family  $\mathcal{F}$  satisfies the following properties:

$$|\mathcal{F}| \leq 2p \binom{n}{i_0}^k \leq 2n^{i_0k}p = \epsilon n \quad \text{and} \quad |\mathcal{F}_i \cap \mathcal{F}| \geq \frac{p}{2} \cdot \lambda n^{i_0k} = \frac{1}{4} \lambda \epsilon n \text{ for any } i \in [t].$$

Furthermore, the expected number of intersecting pairs of members in  $\mathcal{F}$  is at most

$$p^2 n^{i_0k} \cdot i_0k \cdot n^{i_0k-1} = \epsilon^2 i_0k n / 4.$$

By Markov's inequality,  $\mathcal{F}$  contains at most  $\epsilon^2 i_0k n / 2$  intersecting pairs of  $i_0k$ -sets with probability at least  $1/2$ .

Let  $\mathcal{F}' \subset \mathcal{F}$  be the subfamily obtained by deleting one  $i_0k$ -set from each intersecting pair and removing all  $i_0k$ -sets that do not belong to any  $\mathcal{F}_i$ ,  $i \in [t]$ . Therefore,  $|\mathcal{F}'| \leq |\mathcal{F}| \leq \epsilon n$  and for each  $i \in [t]$ , we have

$$|\mathcal{F}_i \cap \mathcal{F}'| \geq |\mathcal{F}_i \cap \mathcal{F}| - \frac{1}{2} \epsilon^2 i_0k n \geq \frac{1}{4} \lambda \epsilon n - \frac{1}{2} \epsilon^2 i_0k n = \frac{\lambda^2}{32 i_0k} n$$

and we are done.  $\square$

Now we prove our first absorbing lemma, which is an analog of [23, Fact 2.3] for  $k$ -partite  $k$ -graphs.

**Lemma 2.2** (Absorbing lemma I). *Given  $0 < \alpha \ll \epsilon$ , the following holds for sufficiently large  $n$ . Let  $H$  be a  $k$ -partite  $k$ -graph with parts of size  $n$  such that  $\delta_{[k] \setminus \{i\}}(H) \geq \epsilon n$  for  $i \in [3]$ , then there exists a matching  $M'$  in  $H$  of size at most  $\sqrt{\alpha n}$  such that for every balanced  $2k$ -set  $S$  of  $H$ , the number of  $S$ -absorbing edges in  $M'$  is at least  $\alpha n$ .*

*Proof.* We show that for each balanced  $2k$ -set  $S$ , there are many  $S$ -absorbing edges. Since there are at most  $n^{2k}$  balanced  $2k$ -sets, the existence of the desired matching follows from Proposition 2.1.

Given  $0 < \alpha \ll \epsilon$  and sufficiently large  $n$ , let  $H$  be a  $k$ -partite  $k$ -graph with parts of size  $n$  such that  $\delta_{[k] \setminus \{i\}}(H) \geq \epsilon n$  for  $i \in [3]$ . We claim that for every balanced  $2k$ -set  $S$ , the number of  $S$ -absorbing edges is at least  $\epsilon^3 n^k / 2$ . Indeed, assume that  $\{w, v\} := S \cap V_3$  and  $u \in S \cap V_2$ . We obtain  $S$ -absorbing edges  $e = \{v_1, v_2, \dots, v_k\}$  as follows.

First, for each  $j \in [4, k]$ , we choose arbitrary  $v_i \in V_j \setminus S$  – there are  $n - 2$  choices for each  $v_j$ . Having selected  $\{v_4, v_5, \dots, v_k\}$ , we select a neighbor of  $\{u, v, v_4, \dots, v_k\}$  as  $v_1$ . Next, we choose a neighbor of  $S'$  as  $v_2$ , where  $S'$  is an arbitrary crossing  $(k-1)$ -subset of  $S \setminus V_2$  that contains  $w$ . Finally, we choose a neighbor of  $\{v_4, \dots, v_k, v_1, v_2\}$  as  $v_3$ . There are at least  $\epsilon n - 2$  choices for  $v_j$  for  $j = 1, 2, 3$ . Hence, there are at least

$$(n-2)^{k-3} (\epsilon n - 2)^3 \geq \frac{1}{2} \epsilon^3 n^k > \sqrt{32k\alpha} n^k$$

$S$ -absorbing edges, since  $n$  is sufficiently large and  $\alpha \ll \epsilon$ . Then we get the absorbing matching  $M'$  by applying Proposition 2.1 with  $\lambda = \sqrt{32k\alpha}$  and  $i_0 = 1$ .  $\square$

Our second absorbing lemma deals with the case when only two partite minimum codegrees are large and their sum is about  $n$ .

**Lemma 2.3** (Absorbing Lemma II). *Given  $0 < \epsilon \ll 1/k$ , there exists  $t \in \mathbb{N}$  and  $0 < \alpha \ll \gamma, 1/t$  such that the following holds for sufficiently large integer  $n$ . Let  $H$  be a  $k$ -partite  $k$ -graph with parts of size  $n$  and  $a_i := \delta_{[k] \setminus \{i\}}(H)$  for each  $i \in [k]$ . If  $\sum_{i \in [k]} a_i \geq (1 - \epsilon)n$ ,  $a_1 \geq a_2 \geq \epsilon n$  and  $a_j < \epsilon n$  for  $j \geq 3$ , then one of the following holds.*

- (i)  $H$  is  $2k^2\epsilon$ - $D$ -extremal.
- (ii) There exists a family of disjoint  $tk$ -sets  $\mathcal{F}'$  in  $H$  of size  $|\mathcal{F}'| \leq \sqrt{\alpha n}$  such that each  $F \in \mathcal{F}'$  spans a matching of size  $t$  and for every crossing  $k$ -set  $S$  of  $H$ , the number of  $S$ -perfect-absorbing sets in  $\mathcal{F}'$  is at least  $\alpha n$ .

The proof of Lemma 2.3 is more involved than that of Lemma 2.2 – we need to apply a *lattice-based absorbing method*, a variant of the absorbing method developed recently by the first author [6]. Roughly speaking, the method provides a vertex partition  $\mathcal{P}$  of  $H$  (Lemma 2.6) which refines the original  $k$ -partition so that we can work on the vectors of  $\{0, 1\}^{|\mathcal{P}|}$  that represent the edges of  $H$ . Using the information on these

vectors, we will show that if Lemma 2.3 (ii) does not hold, then  $H$  is close to a divisibility barrier based on  $\mathcal{P}$ . The rest of this section is devoted to the proof of Lemma 2.3, for which we need the following notation and auxiliary results.

The following concepts are introduced by Lo and Markström [16]. Let  $H$  be a  $k$ -partite  $k$ -graph with  $n$  vertices in each part. Given  $\beta > 0$ ,  $i \in \mathbb{N}$ ,  $j \in [k]$  and two vertices  $u, v \in V_j$ , we say that  $u, v$  are  $(\beta, i)$ -reachable in  $H$  if and only if there are at least  $\beta n^{ik-1}$   $(ik-1)$ -sets  $W$  such that both  $H[\{u\} \cup W]$  and  $H[\{v\} \cup W]$  contain perfect matchings. In this case  $W$  is called a *reachable set* for  $u, v$ . If all  $u, v \in V_j$  are  $(\beta, i)$ -reachable in  $H$ , then we say  $V_j$  is  $(\beta, i)$ -closed in  $H$ . Denote by  $\tilde{N}_{\beta, i}(v)$  the set of vertices that are  $(\beta, i)$ -reachable to  $v$  in  $H$ . Clearly, since  $H$  is  $k$ -partite, for any  $j \in [k]$  and  $v \in V_j$ ,  $\tilde{N}_{\beta, i}(v) \subseteq V_j$ .

**Fact 2.4.** *Let  $H$  be a  $k$ -partite  $k$ -graph with parts of size  $n$ . Let  $a_1 := \delta_{[k] \setminus \{1\}}(H)$ .*

- (i) *For any  $i \in [2, k]$  and  $v \in V_i$ , we have  $\deg(v) \geq a_1 n^{k-2}$ .*
- (ii) *If  $a_1 \geq (1/3 + \gamma)n$ , then for any  $i \in [2, k]$ , any set of three vertices  $u, v, w \in V_i$  contains a pair of vertices which are  $(\gamma, 1)$ -reachable.*

*Proof.* To see (i), note that we can obtain an edge containing  $v$  by first choosing a  $(k-2)$ -set  $S \in \Pi_{j \neq 1, i} V_j$ , and then choosing a neighbor of  $\{v\} \cup S$ . To see (ii), by (i) and  $a_1 \geq (1/3 + \gamma)n$ , we have  $|N(u)|, |N(v)|, |N(w)| \geq (1/3 + \gamma)n^{k-1}$ . Also note that  $|N(u) \cup N(v) \cup N(w)| \leq n^{k-1}$ , then by the inclusion-exclusion principle, we have

$$n^{k-1} \geq |N(u)| + |N(v)| + |N(w)| - |N(u) \cap N(v)| - |N(u) \cap N(w)| - |N(v) \cap N(w)|.$$

So we get  $|N(u) \cap N(v)| + |N(u) \cap N(w)| + |N(v) \cap N(w)| \geq 3\gamma n^{k-1}$ . Without loss of generality, assume that  $|N(u) \cap N(v)| \geq \gamma n^{k-1}$ . This implies that  $u$  and  $v$  are  $(\gamma, 1)$ -reachable.  $\square$

**Proposition 2.5.** *Suppose  $0 < 1/n \ll \epsilon \ll 1/k$  and let  $H$  be a  $k$ -partite  $k$ -graph with  $n$  vertices in each part such that  $\delta_{[k] \setminus \{1\}}(H), \delta_{[k] \setminus \{2\}}(H) \geq (1/2 - \epsilon)n$ . Then for any  $j = 1, 2$  and  $v \in V_j$ ,  $|\tilde{N}_{\epsilon/3, 1}(v)| \geq (1/2 - 2\epsilon)n$ . Moreover, for each  $j \in [3, k]$ , either  $|\tilde{N}_{\epsilon/3, 1}(v)| \geq \epsilon n$  holds for all vertices  $v \in V_j$ , or there exists a set  $V'_j \subseteq V_j$  of size at most  $\epsilon n + 1$  such that  $V_j \setminus V'_j$  is  $(\epsilon/3, 1)$ -closed in  $H$ .*

*Proof.* Fix a vertex  $v \in V_j$  for some  $j = 1, 2$ , note that for any other vertex  $u \in V_j$ ,  $u \in \tilde{N}_{\epsilon/3, 1}(v)$  if and only if  $|N(u) \cap N(v)| \geq \epsilon n^{k-1}/3$ . By double counting, we have

$$\sum_{S \in N(v)} \deg(S) < |\tilde{N}_{\epsilon/3, 1}(v)| \cdot |N(v)| + n \cdot \epsilon n^{k-1}/3.$$

We know that  $\sum_{S \in N(v)} \deg(S) \geq (1/2 - \epsilon)n|N(v)|$ . Moreover, since  $v$  is not in  $V_1$  (or  $V_2$ ), by Fact 2.4 (i),  $|N(v)| \geq (1/2 - \epsilon)n^{k-1} > n^{k-1}/3$ . Thus,

$$|\tilde{N}_{\epsilon/3, 1}(v)| > \left(\frac{1}{2} - \epsilon\right)n - \frac{\epsilon n^k/3}{|N(v)|} \geq \left(\frac{1}{2} - \epsilon\right)n - \epsilon n = \left(\frac{1}{2} - 2\epsilon\right)n.$$

Now assume  $j \in [3, k]$  and assume that  $|\tilde{N}_{\epsilon/3, 1}(v)| < \epsilon n$  for some  $v \in V_j$ . Let  $V'_j := \{v\} \cup \tilde{N}_{\epsilon/3, 1}(v)$ . Thus  $|V'_j| \leq \epsilon n + 1$ . For any  $u, u' \in V_j \setminus V'_j$ , since  $u \notin \tilde{N}_{\epsilon/3, 1}(v)$  and  $u' \notin \tilde{N}_{\epsilon/3, 1}(v)$ , by Fact 2.4 (ii), we conclude that  $u$  and  $u'$  are  $(\epsilon/3, 1)$ -reachable. This implies that  $V_j \setminus V'_j$  is  $(\epsilon/3, 1)$ -closed.  $\square$

**Lemma 2.6.** [10, Lemma 6.3] *Given  $c \in \mathbb{N}$  and  $0 < \beta \ll \gamma \ll 1/c, \delta'$ , the following holds for sufficiently large  $m$ . Suppose that  $H$  is an  $m$ -vertex  $k$ -graph, and a vertex set  $S \subseteq V(H)$  satisfies that  $|\tilde{N}_{\gamma, 1}(v)| \geq \delta' m$  for any  $v \in S$  and every set of  $c+1$  vertices in  $S$  contains at least two vertices that are  $(\gamma, 1)$ -reachable. Then we can find a partition  $\mathcal{P}_0$  of  $S$  into  $V_1, \dots, V_d$  with  $d \leq \min\{\lfloor 1/\delta' \rfloor, c\}$  such that for any  $i \in [d]$ ,  $|V_i| \geq (\delta' - \gamma)m$  and  $V_i$  is  $(\beta, 2^{c-1})$ -closed in  $H$ .*

**Proposition 2.7.** [16, Proposition 2.1] *For  $\epsilon, \beta > 0$  and integer  $i \geq 1$ , there exists  $\beta_0 > 0$  and an integer  $n_0$  satisfying the following. Suppose  $H$  is a  $k$ -graph of order  $n \geq n_0$  and there exists a vertex  $x \in V(H)$  with  $|\tilde{N}_{\beta, i}(x)| \geq \epsilon n$ . Then for all  $0 < \beta' \leq \beta_0$ ,  $\tilde{N}_{\beta, i}(x) \subseteq \tilde{N}_{\beta', i+1}(x)$ .*

Let  $H$  be a  $k$ -partite  $k$ -graph with parts of size  $n$ . Suppose that  $\mathcal{P} = \{V_0, V_1, \dots, V_d\}$  is a vertex partition of  $V(H)$  for some integer  $d \geq k$  that refines the original  $k$ -partition of  $H$ . In particular,  $V_0$  is so small that

we only consider the edges not intersecting  $V_0$ . The following concepts were introduced by Keevash and Mycroft [11]. The *index vector* of a subset  $S \subseteq V(H)$  with respect to  $\mathcal{P}$  is the vector

$$\mathbf{i}_{\mathcal{P}}(S) := (|S \cap V_1|, \dots, |S \cap V_d|) \in \mathbb{Z}^d.$$

Given an index vector  $\mathbf{v}$ , we denote by  $\mathbf{v}|_{V_i}$  its value at the coordinate that corresponds to  $V_i$ . For  $\mu > 0$ , define  $I_{\mathcal{P}}^{\mu}(H)$  to be the set of all  $\mathbf{v} \in \mathbb{Z}^d$  such that  $H$  contains at least  $\mu n^k$  edges with index vector  $\mathbf{v}$ ; let  $L_{\mathcal{P}}^{\mu}(H)$  denote the lattice in  $\mathbb{Z}^d$  generated by  $I_{\mathcal{P}}^{\mu}(H)$ . For  $i \in [d]$ , let  $\mathbf{u}_{V_i} \in \mathbb{Z}^r$  be the *unit vector* such that  $\mathbf{u}_{V_i}|_{V_i} = 1$  and  $\mathbf{u}_{V_i}|_{V_j} = 0$  for  $j \neq i$ .

**Lemma 2.8.** [8, Lemma 3.4] *Given  $0 < \mu, \beta \ll \epsilon \ll 1/i_0, 1/k$ , there exist  $0 < \beta' \ll \mu, \beta$  and an integer  $t \geq i_0$  such that the following holds for sufficiently large  $m$ . Suppose  $H$  is an  $m$ -vertex  $k$ -graph and  $\mathcal{P} = \{V_0, V_1, \dots, V_d\}$  is a partition with  $d \leq 2k$  such that  $|V_0| \leq \sqrt{\epsilon}m$  and for any  $i \in [d]$ ,  $|V_i| \geq \epsilon^2 m$  and  $V_i$  is  $(\beta, i_0)$ -closed in  $H$ . If  $\mathbf{u}_{V_i} - \mathbf{u}_{V_j} \in L_{\mathcal{P}}^{\mu}(H)$ , then  $V_i \cup V_j$  is  $(\beta', t)$ -closed in  $H$ .*

The following lemma shows that if  $V_1$  or  $V_2$  is closed then Lemma 2.3 (ii) holds.

**Lemma 2.9.** *Let  $\beta, \epsilon > 0$  and  $i_0 \in \mathbb{N}$ . There exists  $\alpha \ll \beta, \epsilon, 1/i_0$  such that the following holds for sufficiently large  $n$ . Suppose  $H$  is a  $k$ -partite  $k$ -graph with parts each of size  $n$  and  $\delta_{[k] \setminus \{i\}}(H) \geq \epsilon n$  for  $i = 1, 2$ . If  $V_1$  or  $V_2$  is  $(\beta, i_0)$ -closed, then there exists a family of disjoint  $i_0 k$ -sets  $\mathcal{F}'$  in  $H$  of size  $|\mathcal{F}'| \leq \sqrt{\alpha}n$  such that each  $F \in \mathcal{F}'$  spans a matching of size  $t$  and for every crossing  $k$ -set  $S$  of  $H$ , the number of  $S$ -perfect-absorbing sets in  $\mathcal{F}'$  is at least  $\alpha n$ .*

*Proof.* Without loss of generality, assume that  $V_1$  is  $(\beta, i_0)$ -closed, i.e., any  $u, v \in V_1$  are  $(\beta, i_0)$ -reachable.

Fix a crossing  $k$ -set  $S = \{v_1, v_2, \dots, v_k\}$  such that  $v_j \in V_j$ , we claim there are at least  $\sqrt{32i_0 k \alpha} n^{i_0 k}$   $S$ -perfect-absorbing  $i_0 k$ -sets. First of all, we find  $v'_1 \in V_1 \setminus \{v_1\}$  such that  $\{v'_1, v_2, \dots, v_k\}$  spans an edge. Since  $\deg(S \setminus \{v_1\}) \geq \epsilon n$ , there are at least  $\epsilon n - 1$  choices for  $v'_1$ . Since  $V_1$  is  $(\beta, i_0)$ -closed, there are at least  $\beta n^{i_0 k - 1}$  reachable  $(i_0 k - 1)$ -sets  $W$  for  $v_1$  and  $v'_1$ . Among them, at least  $\beta n^{i_0 k - 1} - (k - 1)n^{k - 2} \geq \beta n^{i_0 k - 1} / 2$  reachable  $(i_0 k - 1)$ -sets  $W$  are disjoint from  $S$ . To see that  $\{v'_1\} \cup W$  is an absorbing set, note that  $H[\{v'_1\} \cup W]$  has a perfect matching by the definition of  $W$ , and  $H[\{v'_1\} \cup W \cup S]$  has a perfect matching because  $\{v'_1\} \cup (S \setminus \{v_1\})$  spans an edge and  $H[\{v_1\} \cup W]$  has a perfect matching by the definition of  $W$ . In total, we have at least  $\epsilon \beta n^{i_0 k} / 4 > \sqrt{32i_0 k \alpha} n^{i_0 k}$   $S$ -perfect-absorbing sets. So we get the family of absorbing sets  $\mathcal{F}'$  by applying Proposition 2.1 with  $\lambda = \sqrt{32i_0 k \alpha}$ . Note that each  $F \in \mathcal{F}'$  is an absorbing set for some crossing  $k$ -set  $S$  and thus  $F$  spans a matching of size  $k$ .  $\square$

*Proof of Lemma 2.3.* Given  $\epsilon \ll 1/k$ , pick  $0 < \mu, \beta \ll \epsilon$ . We apply Lemma 2.8 inductively  $k$  times, each time with input  $i_0 := t_{i-1}$ ,  $\beta_{i-1}$  and output  $t_i \in \mathbb{N}$  and  $\beta_i$ , where  $t_0 = 2$ . Then let  $t := t_k$  and reselect the constants such that

$$0 < 1/n \ll \alpha \ll \beta_k \ll \beta_{k-1} \ll \dots \ll \beta_1 \ll \mu, \beta \ll \gamma \ll \epsilon \ll 1/k, 1/t_k.$$

Let  $H$  be a  $k$ -partite  $k$ -graph as given by the lemma. Suppose (ii) does not hold. In particular, by Lemma 2.9, we may assume that neither  $V_1$  nor  $V_2$  is  $(\beta_k, t_k)$ -closed in  $H$ . By Fact 2.4 (i), we have  $\deg(v) \geq a_1 n^{k-2}$  for any  $v \notin V_1$ , and  $\deg(v) \geq a_2 n^{k-2}$  for any  $v \in V_1$ .

First note that if  $a_1 \geq (1/2 + \epsilon)n$ , then for any  $u, v \in V_2$ , we have  $|N(u) \cap N(v)| \geq 2\epsilon n^{k-1}$ , and thus  $V_2$  is  $(2\epsilon, 1)$ -closed. By Proposition 2.7,  $V_2$  is  $(\beta_k, t_k)$ -closed, a contradiction.

So we may assume that  $a_1 < (1/2 + \epsilon)n$ . Thus, we have

$$a_2 \geq \sum_{i \in [k]} a_i - a_1 - (k - 2)\epsilon n \geq (1 - \epsilon)n - (1/2 + \epsilon)n - (k - 2)\epsilon n = (1/2 - k\epsilon)n,$$

i.e.,  $(1/2 - k\epsilon)n \leq a_2 \leq a_1 < (1/2 + \epsilon)n$ . We apply Proposition 2.5 with  $k\epsilon$  in place of  $\epsilon$  and obtain that, using  $\gamma \leq k\epsilon/3$ ,

- (1) for any  $i = 1, 2$  and  $v \in V_i$ ,  $|\tilde{N}_{\gamma, 1}(v)| \geq (1/2 - 2k\epsilon)n$ ,
- (2) for any  $i \in [3, k]$ , either  $|\tilde{N}_{\gamma, 1}(v)| \geq k\epsilon n$  for all vertices  $v \in V_i$ , or there exists a set  $V'_i \subseteq V_i$  of size at most  $k\epsilon n + 1$  such that  $V''_i = V_i \setminus V'_i$  is  $(\gamma, 1)$ -closed in  $H$ .

Since  $a_1, a_2 \geq (1/2 - k\epsilon)n \geq (1/3 + \gamma)n$ , Fact 2.4 (ii) implies that for any  $i \in [k]$ , every set of three vertices of  $V_i$  contains two vertices that are  $(\gamma, 1)$ -reachable in  $H$ . Together with (1), it allows us to apply Lemma 2.6 to  $V_1, V_2$  with  $c = 2$  and  $\delta' = 1/(2k) - 2\epsilon$  and partition each of  $V_1$  and  $V_2$  into at most two parts such that each part is  $(\beta, 2)$ -closed. If  $V_1$  or  $V_2$  is  $(\beta, 2)$ -closed, then it is  $(\beta_k, t_k)$ -closed, a contradiction.

Thus we assume that each of  $V_1$  and  $V_2$  is partitioned into two parts  $V_1 = A_1 \cup B_1$  and  $V_2 = A_2 \cup B_2$  such that  $A_i, B_i$  are  $(\beta, 2)$ -closed, and

$$|A_i|, |B_i| \geq (\frac{1}{2k} - 2\epsilon - \gamma)kn \geq (\frac{1}{2} - 3k\epsilon)n.$$

Without loss of generality, assume that  $|A_1| \leq |B_1|$  and  $|A_2| \leq |B_2|$ .

Let  $I \subseteq [3, k]$  be the set of  $i$  such that  $|\tilde{N}_{\gamma,1}(v)| \geq k\epsilon n$  for all vertices  $v \in V_i$ , and let  $I' \subseteq I$  consist of those  $i \in I$  such that  $V_i$  is not  $(\beta, 2)$ -closed. Because of (2), we now apply Lemma 2.6 to  $V_i$  for  $i \in I'$  with  $c = 2$  and  $\delta' = \epsilon$  and partition  $V_i$  into at most two parts such that each part is of size at least  $(\epsilon - \gamma)kn \geq \epsilon n$  and is  $(\beta, 2)$ -closed. Since  $V_i, i \in I'$ , is not  $(\beta, 2)$ -closed, it must be the case that  $V_i$  is partitioned into two parts, denoted by  $A_i$  and  $B_i$ . Let  $V_0 = \bigcup_{i \in [3, k] \setminus I} V_i'$  and note that  $|V_0| \leq (k-2)(k\epsilon n + 1) \leq k^2\epsilon n$ . Let  $\mathcal{P}_0$  be the partition of  $V(H)$  consisting of  $V_0, V_i''$  for  $i \in [3, k] \setminus I$ ,  $V_i$  for  $i \in I \setminus I'$ , and  $A_i, B_i$  for  $i \in \{1, 2\} \cup I'$ , namely,

$$\mathcal{P}_0 := \{V_0, A_1, B_1, A_2, B_2, \dots\}.$$

By Proposition 2.7, all parts of  $\mathcal{P}_0$  are  $(\beta, 2)$ -closed.

For  $i \in [k]$ , if  $\mathbf{u}_{A_i} - \mathbf{u}_{B_i} \in L_{\mathcal{P}_0}^\mu(H)$ , then we can *merge*  $A_i$  and  $B_i$  by replacing  $A_i$  and  $B_i$  with  $V_i$ . By Lemma 2.8,  $V_i = A_i \cup B_i$  is  $(\beta_1, t_1)$ -closed. We inductively merge  $A_i, B_i$  as long as  $\mathbf{u}_{A_i} - \mathbf{u}_{B_i} \in L_{\mathcal{P}'}^\mu(H)$ , where  $\mathcal{P}'$  represents the intermediate partition after merging parts. After at most  $k-2$  merges, we obtain a partition  $\mathcal{P} = \{V_0, A_1, B_1, A_2, B_2, \dots\}$  such that each part except  $V_0$  is  $(\beta_k, t_k)$ -closed. Let  $\tilde{I} \subseteq [k]$  be the set of  $i$  such that  $A_i \in \mathcal{P}$ . We know that  $\mathbf{u}_{A_i} - \mathbf{u}_{B_i} \notin L_{\mathcal{P}}^\mu(H)$  for any  $i \in \tilde{I}$ . Let  $d$  be the number of the parts in  $\mathcal{P}$  except  $V_0$ .

Let  $A := \bigcup_{i \in \tilde{I}} A_i$  and let  $E' = E(H \setminus V_0)$ . A vector  $\mathbf{v} \in \{0, 1\}^d$  is *even* (respectively, *odd*) if there is an even (respectively, odd) number of  $i \in \tilde{I}$  such that  $\mathbf{v}|_{A_i} = 1$ . An edge  $e \in E'$  is *even* (otherwise *odd*) if its index vector  $\mathbf{i}_{\mathcal{P}}(e)$  is even, namely,  $|e \cap A|$  is even. Write  $T := I_{\mathcal{P}}^\mu(H)$ . We have the following observations, provided that all operations take place in  $\{0, 1\}^d$ .

- (†) If  $\mathbf{w} \in T$ , then  $\mathbf{w} + \mathbf{u}_{A_i} - \mathbf{u}_{B_i} \notin T$  for  $i \in \tilde{I}$ .
- (‡) If  $\mathbf{w} \in T$ , then  $\mathbf{w} + \mathbf{u}_{A_i} - \mathbf{u}_{B_i} + \mathbf{u}_{A_1} - \mathbf{u}_{B_1} \in T$  for  $i \in \tilde{I}$ .

Indeed, for (†), if  $\mathbf{w} \in T$ , then  $\mathbf{w} + \mathbf{u}_{A_i} - \mathbf{u}_{B_i} \notin T$  because  $\mathbf{u}_{A_i} - \mathbf{u}_{B_i} \notin L_{\mathcal{P}}^\mu(H)$  for  $i \in \tilde{I}$ . For (‡), note that  $\mathbf{w} + \mathbf{u}_{A_i} - \mathbf{u}_{B_i}$  has 1 at  $k$  coordinates, which correspond to *nonempty* sets  $C_j \subseteq V_i, j \in [k]$  (where  $C_i$  is  $V_i$  or  $V_i''$  or  $A_i$  or  $B_i$ ), each of size at least  $\epsilon n$ . The number of edges in  $H$  that contain a crossing  $(k-1)$ -set in  $\prod_{j \in [2, k]} C_j$  is at least  $(\epsilon n)^{k-1} a_1$ . Since  $\mathbf{w} + \mathbf{u}_{A_i} - \mathbf{u}_{B_i} \notin T$ , the number of edges  $e$  in  $H$  with  $\mathbf{i}_{\mathcal{P}}(e) = \mathbf{w} + \mathbf{u}_{A_i} - \mathbf{u}_{B_i}$  is less than  $\mu(kn)^k$ . Consequently, the number of edges  $e$  with  $\mathbf{i}_{\mathcal{P}}(e) = \mathbf{w} + \mathbf{u}_{A_i} - \mathbf{u}_{B_i} + \mathbf{u}_{A_1} - \mathbf{u}_{B_1}$  is at least  $(\epsilon n)^{k-1} a_1 - \mu(kn)^k \geq \mu(kn)^k$ , because  $\mu \ll \epsilon$  and  $a_1 \geq n/3$ . Hence  $\mathbf{w} + \mathbf{u}_{A_i} - \mathbf{u}_{B_i} + \mathbf{u}_{A_1} - \mathbf{u}_{B_1} \in T$ . This proves (‡).

We claim that all the vectors in  $T$  have the same parity. Indeed, assume that there is an even vector  $\mathbf{v} \in T$ . By (‡), all even vectors are in  $T$ . Together with (†), this implies that  $T$  does not contain any odd vector.

Assume that  $T$  only contains even vectors (the case that  $T$  only contains odd vectors is analogous). Thus,  $E'$  contains at most  $2^k \mu n^k$  of odd edges. Let  $B := V \setminus A$ , and let  $A := \bigcup A_i$  and  $B := \bigcup B_i$ . Recall that  $(1/2 - 3k\epsilon)n \leq |A_1|, |A_2| \leq n/2$ . Moreover, recall that  $\deg(v) \geq (1/2 - k\epsilon)n^{k-1}$  for all  $v \in V(H)$  and thus  $|E(H)| \geq (1/2 - k\epsilon)n^k$ . The number of even edges in  $E(H)$  is at least

$$|E(H)| - 2^k \mu n^k - |V_0| n^{k-1} \geq (1/2 - k\epsilon)n^k - 2^k \mu n^k - (k-2)(k\epsilon n + 1)n^{k-1} \geq (1/2 - k^2\epsilon)n^k,$$

as  $\mu \ll \epsilon$  and  $|V_0| \leq (k-2)(k\epsilon n + 1)$ . Let  $|A_1| = n/2 - y$  for some  $0 \leq y \leq 3k\epsilon n$  and assume that the number of odd crossing  $(k-1)$ -set in  $V \setminus V_1$  is  $x$  for some  $0 \leq x \leq n^{k-1}$ , then we get

$$\begin{aligned} |E_{\text{even}}(A, B)| &= (n/2 - y)x + (n/2 + y)(n^{k-1} - x) \\ &= n^k/2 + y(n^{k-1} - 2x) \leq n^k/2 + 3k\epsilon n^k. \end{aligned}$$

Thus we have  $|E_{\text{even}}(A, B) \setminus E(H)| \leq n^k/2 + 3k\epsilon n^k - (1/2 - k^2\epsilon)n^k \leq 2k^2\epsilon n^k$ . Together with  $(1/2 - 3k\epsilon)n \leq |A_1|, |A_2| \leq n/2$  and  $(1/2 - k\epsilon)n \leq a_2 \leq a_1 < (1/2 + \epsilon)n$ , we get that  $H$  is  $2k^2\epsilon$ -D-extremal. So (i) holds and we are done.  $\square$

3. NONEXTREMAL  $k$ -PARTITE  $k$ -GRAPHS: PROOF OF THEOREM 1.4

In this section we first show that every  $k$ -partite  $k$ -graph  $H$  contains an almost perfect matching if  $\sum a_i$  is near  $n$  and  $H$  is not close to  $H_0$ . The following lemma is an analog of [7, Lemma 1.7] in  $k$ -partite  $k$ -graphs. To make it applicable to other problems, we prove it under a weaker assumption which allows a small fraction of crossing  $(k-1)$ -sets to have small degree.

**Lemma 3.1** (Almost perfect matching). *For any  $\gamma > 0$ , integer  $k \geq 3$  and  $0 \leq \eta \ll \alpha$ , the following holds for sufficiently large integer  $n$ . For  $i \in [k]$ , let  $a_i = a_i(n)$  such that  $\sum_{i \in [k]} a_i \geq (1-\gamma)n$ . Let  $H$  be a  $k$ -partite  $k$ -graph with parts of size  $n$  which is not  $2\gamma$ -S-extremal. Suppose for each  $i \in [k]$ , there are fewer than  $\eta n^{k-1}$  crossing  $(k-1)$ -sets  $S$  such that  $S \cap V_i = \emptyset$  and  $\deg(S) < a_i$ . Then  $H$  contains a matching that covers all but at most  $\alpha n$  vertices in each vertex class.*

*Proof.* Let  $M$  be a maximum matching of size  $m$  in  $H$ . Let  $V'_i = V_i \cap V(M)$  and  $U_i = V_i \setminus V(M)$ . Suppose to the contrary, that  $|U_1| = \dots = |U_k| > \alpha n$ .

Let  $t = \lceil k(k-1)/\gamma \rceil$ . We find a family  $\mathcal{A}$  of disjoint crossing  $(k-1)$ -subsets  $A_1, \dots, A_{kt}$  of  $V \setminus V(M)$  such that  $A_j \cap V_i = \emptyset$  and  $\deg(A_j) \geq a_i$  whenever  $j \equiv i \pmod k$ . This can be done greedily because when selecting  $A_j$ , the crossing  $(k-1)$ -sets that cannot be picked are those either intersect the ones that have been picked, or those with low degree, whose number is at most

$$k(k-1)tn^{k-2} + \eta n^{k-1} < (\alpha n)^{k-1} < \prod_{\ell \in [k] \setminus \{i\}} |U_\ell|,$$

because  $\eta \ll \alpha$  and  $n$  is sufficiently large. Note that the neighbors of  $A_j$  are in  $V'_i$  with  $j \equiv i \pmod k$  by the maximality of  $M$ .

For  $i \in [k]$ , let  $D_i$  be the set of the vertices of  $V'_i$  that have at least  $k-1$  neighbors in  $\mathcal{A}$  and let  $D = \bigcup D_i$ . We claim that  $|e \cap D| \leq 1$  for each  $e \in M$ . Indeed, otherwise assume that  $x, y \in e \cap D$  and pick  $A_i, A_j$  for some  $i, j \in [t]$  such that  $\{x\} \cup A_i, \{y\} \cup A_j \in E(H)$ . We obtain a matching of size  $m+1$  by deleting  $e$  and adding  $\{x\} \cup A_i$  as well as  $\{y\} \cup A_j$  in  $M$ , contradicting the maximality of  $M$ .

Next we show that  $|D_i| \geq a_i - \gamma n/k$  for each  $i \in [k]$ . Since there are no edges between  $A_j$  and  $V'_i$  for  $j \not\equiv i \pmod k$ , by counting the number of edges between  $V'_i$  and  $\mathcal{A}$ , we get

$$t \cdot a_i \leq \sum_{j \equiv i \pmod k} \deg(A_j) \leq |D_i|t + n(k-1).$$

Since  $t \geq k(k-1)/\gamma$ , it follows that

$$|D_i| \geq a_i - \frac{n(k-1)}{t} \geq a_i - \frac{\gamma n}{k}.$$

Define  $M' := \{e \in M : e \cap D \neq \emptyset\}$ . Then for each  $i \in [k]$ , we have

$$|(V(M') \setminus D) \cap V_i| = \sum_{j \neq i} |D_j| \geq \sum_{j \in [k]} (a_j - \frac{\gamma n}{k}) - a_i \geq n - a_i - 2\gamma n.$$

Since  $H$  is not  $2\gamma$ -S-extremal,  $H[V(M') \setminus D]$  contains at least one edge, denoted by  $e_0$ . Note that  $e_0 \notin M$  because each edge of  $M'$  contains exactly one vertex of  $D$  and  $e_0 \subset V(M') \setminus D$ . Assume that  $e_0$  intersects  $e_1, \dots, e_p$  in  $M$  for some  $2 \leq p \leq k$ . Suppose  $\{v_j\} := e_j \cap D$ . Note that  $v_j \notin e_0$  for all  $j \in [p]$ . Since each  $v_j$  has at least  $k-1$  neighbors in  $\mathcal{A}$ , we can greedily pick  $A_{\ell_1}, \dots, A_{\ell_p} \in \mathcal{A}$  such that  $\{v_j\} \cup A_{\ell_j} \in E(H)$  for all  $j \in [p]$ . Let  $M''$  be the matching obtained from  $M$  after replacing  $e_1, \dots, e_p$  by  $e_0$  and  $\{v_j\} \cup A_{\ell_j}$  for  $j \in [p]$ . Thus,  $M''$  has  $m+1$  edges, contradicting the choice of  $M$ .  $\square$

Now we are ready to prove Theorem 1.4.

*Proof of Theorem 1.4.* Given  $\gamma \ll \epsilon \ll 1/k$ , let  $t \in \mathbb{N}$  and  $0 < \alpha \ll \gamma, 1/t$  be returned by Lemma 2.3. Let  $n \in \mathbb{N}$  be sufficiently large. Suppose  $H$  is a  $k$ -partite  $k$ -graph with parts of size  $n$ , which is not  $\gamma$ -S-extremal. Let  $a_i := \delta_{[k] \setminus \{i\}}(H)$  for  $i \in [k]$ . Suppose  $(1-\epsilon)n \geq a_1 \geq \dots \geq a_k$ ,  $a_2 \geq \epsilon n$ , and  $\sum_{i \in [k]} a_i \geq (1-\gamma/5)n$ . Moreover, suppose (ii) and (iii) fail and we will show that (i) holds.

First assume that  $a_1 \geq a_2 \geq a_3 \geq \epsilon n$ . We first apply Lemma 2.2 on  $H$  and find the absorbing matching  $M'$  of size at most  $\sqrt{\alpha n}$  such that for every balanced  $2k$ -set  $S \subset V(H)$ , the number of  $S$ -absorbing edges in  $M'$  is at least  $\alpha n$ .



Let  $H' := H \setminus V(M')$ ,  $n' := |V(H') \cap V_i| \geq (1 - \sqrt{\alpha})n$  and  $a'_i := \delta_{[k] \setminus \{i\}}(H')$ . Note that  $\sum_{i \in [k]} a'_i \geq \sum_{i \in [k]} a_i - k\sqrt{\alpha}n \geq (1 - 2\gamma/5)n'$ . If  $H'$  is  $(4\gamma/5)$ -S-extremal, i.e.,  $V(H')$  contains an independent set  $I$  such that  $|I \cap (V_i \cap V(H'))| \geq n' - a'_i - 4\gamma n'/5$  for each  $i \in [k]$ , then we get that  $H$  is  $\gamma$ -S-extremal since

$$n' - a'_i - 4\gamma n'/5 \geq (1 - \sqrt{\alpha})n - a_i - 4\gamma n/5 \geq n - a_i - \gamma n,$$

a contradiction. Thus,  $H'$  is not  $(4\gamma/5)$ -S-extremal. By applying Lemma 3.1 on  $H'$  with parameters  $2\gamma/5$ ,  $\alpha$  and  $\eta = 0$ , we obtain a matching  $M''$  in  $H'$  that covers all but at most  $\alpha n$  vertices in each vertex class.

Since every balanced  $2k$ -set  $S \subset V(H)$  has at least  $\alpha n$   $S$ -absorbing edges in  $M'$ , we can repeatedly absorb the leftover vertices until there is one vertex left in each class. Denote by  $\tilde{M}$  the matching obtained after absorbing the leftover vertices into  $M'$ . Therefore  $\tilde{M} \cup M''$  is the required matching of size  $n - 1$  in  $H$ .

Second assume that  $a_1 \geq a_2 \geq \epsilon n$  and  $a_i < \epsilon n$  for  $i \in [3, k]$ . Since (iii) does not hold, by applying Lemma 2.3, there exists a family of disjoint absorbing  $tk$ -sets  $\mathcal{F}'$  of size  $|\mathcal{F}'| \leq \sqrt{\alpha}n$  such that each  $F \in \mathcal{F}'$  has a perfect matching and for every crossing  $k$ -set  $S$  of  $H$ , the number of  $S$ -perfect-absorbing sets in  $\mathcal{F}'$  is at least  $\alpha n$ .

Let  $H' := H \setminus V(M')$  and  $n' := |V(H') \cap V_i| \geq (1 - t\sqrt{\alpha})n$  and  $a'_i := \delta_{[k] \setminus \{i\}}(H')$ . Note that  $\sum_{i \in [k]} a'_i \geq \sum_{i \in [k]} a_i - kt\sqrt{\alpha}n \geq (1 - 2\gamma/5)n'$  as  $\alpha \ll \epsilon, 1/t$ . If  $H'$  is  $(4\gamma/5)$ -S-extremal, then  $H$  is  $\gamma$ -S-extremal similarly as in the last case, a contradiction. Assume that  $H'$  is not  $(4\gamma/5)$ -S-extremal. By applying Lemma 3.1 on  $H'$  with parameters  $2\gamma/5$ ,  $\alpha$  and  $\eta = 0$ , we obtain a matching  $M''$  in  $H'$  that covers all but at most  $\alpha n$  vertices in each vertex class. Let  $U$  be the set of leftover vertices. Since any crossing  $k$ -subset  $S$  of  $U$  has at least  $\alpha n$   $S$ -perfect-absorbing  $tk$ -sets in  $\mathcal{F}'$ , we can greedily absorb all the leftover vertices into  $\mathcal{F}$ . Denote by  $\tilde{M}$  the resulting matching that covers  $V(\mathcal{F}') \cup U$ . We obtain a perfect matching  $\tilde{M} \cup M''$  of  $H$ .  $\square$

#### 4. PROOF OF THEOREM 1.5

We prove Theorem 1.5 in this section. Following the approach in [7], we use the following weaker version of a result from Pikhurko [19]. Let  $H$  be a  $k$ -partite  $k$ -graph with parts  $V_1, \dots, V_k$ . Given  $L \subseteq [k]$ , recall that

$$\delta_L(H) = \min \left\{ \deg(S) : S \in \prod_{i \in L} V_i \right\}.$$

**Lemma 4.1.** [19, Theorem 3] *Given  $k \geq 2$  and  $L \subseteq [k]$ , let  $m$  be sufficiently large. Let  $H$  be a  $k$ -partite  $k$ -graph with parts  $V_1, \dots, V_k$  of size  $m$ . If*

$$\delta_L(H)m^{|L|} + \delta_{[k] \setminus L}(H)m^{k-|L|} \geq \frac{3}{2}m^k,$$

*then  $H$  contains a perfect matching.*

*Proof of Theorem 1.5.* Let  $\gamma \ll \epsilon$ ,  $\alpha = \sqrt{\gamma}$ , and  $n$  be sufficiently large. Suppose that  $H$  is a  $k$ -partite  $k$ -graph with parts  $V_1, \dots, V_k$  each of size  $n$ . Let  $a_i := \delta_{[k] \setminus \{i\}}(H)$  for  $i \in [k]$ . Suppose that  $(1 - \epsilon)n \geq a_1 \geq a_2 \geq \dots \geq a_k$  and  $a_2 \geq \epsilon n$ . Assume that (ii) does not hold. Our goal is to find a matching in  $H$  of size at least  $\min\{n - 1, \sum_{i \in [k]} a_i\}$ .

We may assume that  $\sum_{i \in [k]} a_i \geq n - k + 3$  otherwise we are done by Fact 1.3. If  $\sum_{i \in [k]} a_i \geq n + k - 1$ , then we repeatedly remove edges from  $H$  until  $\sum_{i \in [k]} a_i \leq n - k + 2$ . Since removing one edge may decrease  $\sum_{i \in [k]} a_i$  by at most  $k$ , it is guaranteed that  $\sum_{i \in [k]} a_i \geq n - 1$  throughout the process. Thus,  $\min\{n - 1, \sum_{i \in [k]} a_i\} = n - 1$  throughout the process. We thus assume that

$$(4.1) \quad n - k + 3 \leq \sum_{i \in [k]} a_i \leq n + k - 2.$$

Assume that  $H$  is  $\gamma$ -S-extremal, namely, there is an independent set  $C \subseteq V(H)$  such that  $|C \cap V_i| \geq n - a_i - \gamma n$  for each  $i \in [k]$ . Let  $C_i := C \cap V_i$ . We know that  $|C_i| \geq n - a_i - \gamma n \geq (\epsilon - \gamma)n$  from our assumption. For each  $i \in [k]$ , we partition each  $V_i \setminus C_i$  into  $A_i \cup B_i$  such that

$$(4.2) \quad A_i := \left\{ x \in V_i \setminus C_i : \deg(x, C) \geq (1 - \alpha) \prod_{j \neq i} |C_j| \right\},$$

and  $B_i := V_i \setminus (A_i \cup C_i)$ . Moreover, let  $A := \bigcup_{1 \leq i \leq k} A_i$  and  $B := \bigcup_{1 \leq i \leq k} B_i$ .

**Claim 4.2.** For  $i \in [k]$ , we have

- (1)  $a_i \leq |A_i| + |B_i| \leq a_i + \gamma n$ ,
- (2)  $|B_i| \leq \alpha n$ , and
- (3)  $a_i - \alpha n \leq |A_i| \leq a_i + \gamma n$ .

*Proof.* For  $i \in [k]$ , since  $|C_i| \geq n - a_i - \gamma n$ , we have  $|A_i| + |B_i| \leq a_i + \gamma n$ . For any crossing  $(k-1)$ -set  $S \subset C \setminus V_i$ , we have  $N(S) \subseteq A_i \cup B_i$ . By the codegree condition, we have  $|A_i| + |B_i| \geq a_i$ .

Let  $E(A_i \cup B_i, C)$  denote the set of the edges that consist of a  $(k-1)$ -set in  $\prod_{j \neq i} C_j$  and one vertex in  $A_i \cup B_i$ . By the definition of  $A_i$ , we have

$$a_i \prod_{j \neq i} |C_j| \leq |E(A_i \cup B_i, C)| \leq |B_i|(1 - \alpha) \prod_{j \neq i} |C_j| + |A_i| \prod_{j \neq i} |C_j|,$$

which implies  $a_i \leq |A_i| + |B_i| - \alpha|B_i|$ . It follows that  $\alpha|B_i| \leq |A_i| + |B_i| - a_i \leq \gamma n$  by Part (1). Since  $\alpha = \sqrt{\gamma}$ , it follows that  $|B_i| \leq \alpha n$ .

Part (3) follows from Parts (1) and (2) immediately.  $\square$

Our procedure towards the desired matching consists of three steps. First, we remove a matching that covers all the vertices of  $B$ . Second, we remove another matching in order to have  $|C''_i| - \sum_{j \neq i} |A''_j| \leq \max\{1, n - \sum_{i \in [k]} a_i\}$  for all  $i \in [k]$ , where  $C''_i$  and  $A''_i$  denote the set of the remaining vertices in  $C_i$  and  $A_i$ , respectively. Finally, we apply Lemma 4.1 to get a matching that covers all but at most  $\max\{1, n - \sum_{i \in [k]} a_i\}$  vertices in each  $V_i$ .

*Step 1. Cover the vertices of  $B$ .*

For  $i \in [k]$ , define  $t_i := \max\{0, a_i - |A_i|\}$ . By Claim 4.2 (1), we have  $|B_i| \geq a_i - |A_i|$ . Together with the definition of  $t_i$  and Claim 4.2 (2), we have

$$(4.3) \quad t_i \leq |B_i| \leq \alpha n \quad \text{and} \quad t_i + |A_i| \geq a_i.$$

First we build a matching  $M_1^i$  of size  $t_i$  for each  $i \in [k]$  and let  $M_1$  be the union of them. If  $t_i = 0$ , then  $M_1^i = \emptyset$ . Otherwise, since  $a_i = \delta_{[k] \setminus \{i\}}(H)$  and  $C$  is independent, every  $(k-1)$ -set in  $\prod_{j \neq i} C_j$  has at least  $a_i - |A_i| = t_i$  neighbors in  $B_i$ . We greedily pick  $t_i$  disjoint edges each of which consists of a  $(k-1)$ -set in  $\prod_{j \neq i} C_j$  and one vertex in  $B_i$ .

Next for each  $i$ , we greedily build a matching  $M_2^i$  that covers all the remaining vertices in  $B_i$  and let  $M_2$  be the union of them. Indeed, for each of the remaining vertices  $v \in B_i$  with  $i \neq 1$ , we pick one uncovered  $(k-2)$ -set  $S'$  in  $\prod_{j \neq i, 1} C_j$ , and one uncovered vertex in  $N(\{v\} \cup S') \subseteq V_1$ . For each of the remaining vertices in  $v \in B_1$ , we pick one uncovered  $(k-2)$ -set  $S'$  in  $\prod_{j \neq 1, 2} C_j$ , and one uncovered vertex in  $N(\{v\} \cup S') \subseteq V_2$ . Since the number of vertices in  $V_i$  covered by the existing matchings is at most  $|M_1 \cup M_2| \leq |B| \leq k\alpha n < \epsilon n \leq a_2 \leq a_1$ , we can always find an uncovered vertex from  $N(\{v\} \cup S')$ .

For  $i \in [k]$ , let

$$A'_i := A_i \setminus V(M_1 \cup M_2), \quad C'_i := C_i \setminus V(M_1 \cup M_2) \quad \text{and} \quad V'_i := V_i \setminus V(M_1 \cup M_2).$$

*Step 2. Adjust the sizes of  $A'_i$  and  $C'_i$ .*

In this step, we will build a small matching  $M_3$  in order to adjust the sizes of  $A'_i$  and  $C'_i$ .

**Claim 4.3.** There exists a matching  $M_3$  of size at most  $k\gamma n + k$  in  $H[\bigcup_{i=1}^k V'_i]$  so that  $|C'_i \setminus V(M_3)| - \sum_{j \neq i} |A'_j \setminus V(M_3)| = r$  for some integer  $0 \leq r \leq \max\{1, n - \sum_{i \in [k]} a_i\}$ .

*Proof.* Let  $n' := |V'| = |A'_i| + |C'_i|$ . Let  $s_0 := |C'_i| - \sum_{j \neq i} |A'_j| = n' - \sum_{j=1}^k |A'_j|$ , which is independent of  $i$ . We claim that  $-k\gamma n - k \leq s_0 \leq n - \sum_{i \in [k]} a_i$ . Indeed,

$$s_0 \geq (n - |M_1 \cup M_2|) - |A| \geq n - |B| - |A| \stackrel{\text{Claim 4.2}}{\geq} n - \sum_{i \in [k]} (a_i + \gamma n) \stackrel{(4.1)}{\geq} -k\gamma n - k.$$

On the other hand, since  $V(M_1) \cap A = \emptyset$  and  $|M_1| = \sum_{j \in [k]} t_j$ , we have

$$s_0 \leq n - (|M_1| + |M_2|) - \left( \sum_{j \in [k]} |A_j| - |M_2| \right) = n - \sum_{j \in [k]} (t_j + |A_j|) \stackrel{(4.3)}{\leq} n - \sum_{j \in [k]} a_j.$$

If  $s_0 \geq 0$ , then set  $M_3 = \emptyset$  and we are done. Otherwise, we build  $M_3$  by adding edges that contain two or three vertices of  $A$  one by one until  $s \in \{0, 1\}$ , where  $s := (n' - |M_3|) - \sum |A'_j \setminus V(M_3)|$ . Since  $s_0 \geq -k\gamma n - k$  and adding an edge to  $M_3$  increases  $s$  by one or two, we have  $|M_3| \leq k\gamma n + k$ .

Now we show how to build  $M_3$ . First assume that  $a_3 \geq 2k\alpha n$ . In this case we greedily choose the edges of  $M_3$  until  $s \in \{0, 1\}$  by picking two uncovered vertices, one from  $A'_2$  and one from  $A'_3$ , an uncovered  $(k-3)$ -set in  $\prod_{j \in [4, k]} C'_j$ , and one uncovered vertex in  $V'_1$  by the degree condition. To see why we can find these edges, first, we can always pick two uncovered vertices in  $A'_2 \cup A'_3$  because by Claim 4.2 (3),

$$(4.4) \quad |A'_i| \geq |A_i| - |M_2| \geq a_i - \alpha n - k\alpha n \geq k\gamma n + k$$

for  $i = 2, 3$ . Second, we can find an uncovered  $(k-3)$ -set in  $\prod_{j \in [4, k]} C'_j$  because

$$(4.5) \quad |C'_j| \geq |C_j| - |M_1 \cup M_2| \geq \epsilon n - \gamma n - k\alpha n \geq k\gamma n + k.$$

Third, we can find the desired vertex in  $V_1$  because the number of covered vertices in  $V_1$  is at most  $|B| + k\gamma n + k \leq 2k\alpha n < a_1$ .

Next assume that  $|A_1| \geq (1/2 + \epsilon)n$ . In this case we greedily choose the edges of  $M_3$  until  $s \in \{0, 1\}$  by picking an uncovered vertex in  $A'_2$ , an uncovered  $(k-2)$ -set in  $\prod_{j \in [3, k]} C'_j$ , and by the degree condition, one uncovered vertex in  $A'_1$ . To see why we can find these edges, first, we can pick an uncovered  $(k-1)$ -set  $S \in A'_2 \times \prod_{i \in [3, k]} C'_i$  because of (4.4) and (4.5). Second, note that  $a_1 \geq |A_1| - \gamma n \geq (1/2 + \epsilon - \gamma)n$  and

$$|A'_1| \geq |A_1| - |M_2| \geq (\frac{1}{2} + \epsilon)n - k\alpha n = (\frac{1}{2} + \epsilon - k\alpha)n.$$

Thus,  $S$  has at least  $a_1 - (n - |A'_1|) \geq k\gamma n + k$  neighbors in  $A'_1$  so we can find an uncovered neighbor of  $S$ .

Now we assume that  $|A_1| < (1/2 + \epsilon)n$  and  $a_3 < 2k\alpha n$ . In this case we show that (ii) holds. First,  $a_1 \leq |A_1| + \alpha n < (1/2 + \epsilon + \alpha)n$ . Since  $a_i \leq a_3$  for  $i \in [3, k]$  and  $\sum_{i \in [k]} a_i \geq n - k + 3$ , we have

$$a_2 \geq \sum_{i \in [k]} a_i - a_1 - (k-2)a_3 \geq n - k + 3 - (\frac{1}{2} + \epsilon + \alpha)n - 2k(k-2)\alpha n \geq (\frac{1}{2} - 2\epsilon)n,$$

i.e.,  $(1/2 - 2\epsilon)n \leq a_2 \leq a_1 \leq (1/2 + 2\epsilon)n$ . By Claim 4.2 (3),  $|A_2| \leq (1/2 + 2\epsilon)n + \gamma n \leq (1/2 + 3\epsilon)n$  and  $|A_i| \leq a_i + \gamma n \leq 3k\alpha n$  for  $i \in [3, k]$ . The lower bounds on  $a_1, a_2$  implies that  $|A_1|, |A_2| \geq (1/2 - 2\epsilon)n - \alpha n \geq (1/2 - 3\epsilon)n$ . Finally, let  $x$  be the number of crossing  $k$ -sets in  $V(H)$  that intersect  $A_i$  for some  $i \in [3, k]$  and thus  $x \leq (k-2)3k\alpha n \cdot n^{k-1} \leq 3k^2\alpha n^k$ . Let  $y_1$  be the number of non-edges in  $H[A_1, B_2 \cup C_2, \dots, B_k \cup C_k]$  and let  $y_2$  be the number of non-edges in  $H[B_1 \cup C_1, A_2, B_3 \cup C_3, \dots, B_k \cup C_k]$ . By the definition of  $A$  and  $|B_1|, |B_2| \leq \alpha n$ , for  $i = 1, 2$  we have

$$y_i \leq |A_i| \cdot \alpha \prod_{j \neq i} |C_j| + |B_i| \cdot n^{k-1} \leq 2\alpha n^k.$$

Note that  $|E_{\text{odd}}(A, B \cup C) \setminus E(H)| \leq x + y_1 + y_2 \leq 3k^2\alpha n^k + 2 \cdot 2\alpha n^k \leq \epsilon n^k$ . So (ii) holds, a contradiction.  $\square$

*Step 3. Cover the remaining vertices.*

For each  $i \in [k]$ , let

$$A''_i := A'_i \setminus V(M_3), C''_i := C'_i \setminus V(M_3) \text{ and } V''_i := V'_i \setminus V(M_3).$$

By the definitions of  $M_1, M_2, M_3$ , we have  $|M_1 \cup M_2 \cup M_3| \leq k\alpha n + k\gamma n + k \leq (k+1)\alpha n$ . By Claim 4.2 (3), for each  $i \in [k]$ , we have

$$|A''_i| \geq |A_i| - |M_1 \cup M_2 \cup M_3| \geq (a_i - \alpha n) - (k+1)\alpha n \geq a_i - 2k\alpha n.$$

Recall that  $a_1 \geq a_2 \geq \epsilon n$ , by  $\gamma \ll \epsilon$ , we have

$$(4.6) \quad |A''_1|, |A''_2| \geq a_2 - 2k\alpha n \geq \epsilon n/2.$$

By Claim 4.3, we have

$$(4.7) \quad 0 \leq r = |C''_i| - \sum_{j \neq i} |A''_j| \leq \max \left\{ 1, n - \sum_{i \in [k]} a_i \right\}.$$

For  $i \in [k]$ , since  $a_i \leq (1 - \epsilon)n$ , we get that  $|C_i| \geq \epsilon n - \gamma n \geq 2(k+1)\alpha n$ . Thus,

$$(4.8) \quad |C''_i| \geq |C_i| - |M_1 \cup M_2 \cup M_3| \geq |C_i| - (k+1)\alpha n \geq |C_i|/2.$$

Now we greedily match the vertices of  $A''_3, \dots, A''_k$ . Indeed, for any  $3 \leq j \leq k$  and any vertex  $v \in A''_j \subseteq A_j$ , by (4.2), the number of crossing  $(k-1)$ -sets  $S$  in  $\prod_{l \neq j} C''_l$  such that  $S \cup \{v\} \notin E(H)$  is at most

$$\alpha \prod_{l \neq j} |C_l| \leq 2^{k-1} \alpha \prod_{l \neq j} |C''_l|,$$

where we used (4.8). So we can greedily match these vertices because the number of leftover vertices in each  $C''_j$  is at least  $\min\{|A''_1|, |A''_2|\} + r \geq \epsilon n/2$  and thus the number of available crossing  $(k-1)$ -sets is at least  $(\epsilon n/2)^{k-1} \geq 2^{k-1} \alpha n^{k-1} > 2^{k-1} \alpha \prod_{l \neq j} |C''_l|$ . Let  $M_4^0$  be the resulting matching in this step.

Finally, consider the unmatched vertices of  $H$ . Let  $m_i := |A''_i|$  for all  $i = 1, 2$ . Note that the number of unmatched vertices in  $C''_1, C''_2$  are  $m_2 + r$  and  $m_1 + r$ , respectively, and the number of unmatched vertices in  $C''_i$ ,  $i \in [3, k]$  is  $m_1 + m_2 + r$ . For  $i = 1, 2$ , let  $C_1^2$  be a set of  $m_2$  vertices in  $C''_1 \setminus V(M_4^0)$  and let  $C_2^1$  be a set of  $m_1$  vertices in  $C''_2 \setminus V(M_4^0)$ ; for  $i \in [3, k]$ , we can partition all but  $r$  vertices of  $C''_i \setminus V(M_4^0)$  into  $C_i^1$  of size  $m_1$  and  $C_i^2$  of size  $m_2$ . Therefore, we get  $k$ -partite  $k$ -graphs  $H_j := H[A''_j, \bigcup_{\ell \neq j} C_\ell^j]$  for  $j = 1, 2$ . Let us verify the assumptions of Lemma 4.1 for  $H_j$ ,  $j = 1, 2$ .

First, for any  $(k-1)$ -set  $S \in \prod_{\ell \neq j} C_\ell^j$ , the number of its non-neighbors in  $A_j \cup B_j$  is at most

$$|A_j| + |B_j| - a_j \stackrel{\text{Claim 4.2}}{\leq} \gamma n \stackrel{(4.6)}{\leq} \frac{2}{\epsilon} m_j \leq \alpha m_j,$$

as  $\gamma \ll \epsilon$ . So we have

$$\delta_{[k] \setminus \{j\}}(H_j) \geq m_j - \alpha m_j = (1 - \alpha)m_j.$$

Next, for any  $v \in A''_j$ , by (4.2) the number of its non-neighbors in  $\prod_{\ell \neq j} C_\ell^j$  is at most

$$\alpha \prod_{\ell \neq j} |C_\ell^j| < \alpha n^{k-1} \stackrel{(4.6)}{\leq} \alpha \left(\frac{2}{\epsilon} m_j\right)^{k-1} \leq \sqrt{\alpha} m_j^{k-1},$$

which implies that  $\delta_{\{j\}}(H_j) \geq (1 - \sqrt{\alpha})m_j^{k-1}$ . Thus, we have

$$\delta_{\{j\}}(H_j)m_j + \delta_{[k] \setminus \{j\}}(H_j)m_j^{k-1} \geq (1 - \sqrt{\alpha})m_j^{k-1}m_j + (1 - \alpha)m_jm_j^{k-1} > \frac{3}{2}m_j^k,$$

since  $\gamma$  is small enough. By Lemma 4.1, we find a perfect matching  $M_4^j$  in  $H_j$  for each  $j \in [2]$ . Let  $M_4 := M_4^0 \cup M_4^1 \cup M_4^2$ , then  $M_1 \cup M_2 \cup M_3 \cup M_4$  is a matching in  $H$  of size at least  $n - r$ . If  $r \leq 1$ , then we obtain a matching of size at least  $n - 1$ . Otherwise, since  $0 < r \leq n - \sum_{i \in [k]} a_i$ , we get a matching of size at least  $\sum_{i \in [k]} a_i$ .  $\square$

## 5. PROOF OF THEOREM 1.6

We call a binary vector  $\mathbf{v} \in \{0, 1\}^k$  *even* (otherwise *odd*) if it contains an even number of 1. Let  $EV_k$  denote the set of all even vectors in  $\{0, 1\}^k$ . Note that  $|EV_k| = 2^{k-1}$ . Denote by  $(V_1 \cup \dots \cup V_k, E)$  a  $k$ -partite  $k$ -graph with partition  $V_1 \cup \dots \cup V_k$ . Suppose that  $H = (V_1 \cup \dots \cup V_k, E)$  has a vertex bipartition  $A \cup B$ . We call a set  $S \subseteq V(H)$  *even* (or *odd*) if  $|S \cap A|$  is even (or odd). Given a vector  $\mathbf{v} \in \{0, 1\}^k$ , we write  $V^\mathbf{v} = V_1^\mathbf{v} \cup \dots \cup V_k^\mathbf{v}$ , where  $V_i^\mathbf{v} := A_i$  if  $\mathbf{v}|_{V_i} = 1$  and  $V_i^\mathbf{v} := B_i$  otherwise. Let  $H(\mathbf{v}) := H[V^\mathbf{v}]$ . For any crossing  $k$ -set  $S \in V^\mathbf{v}$ , we say that  $\mathbf{v}$  is the *location vector* of  $S$ . Given  $H = (V_1 \cup \dots \cup V_k, E)$ , let  $\deg_H(v) := \prod_{j \neq i} |V_j| - \deg_H(v)$  for  $v \in V_i$  and  $\bar{\delta}_1(H) := \max_{v \in V(H)} \deg_H(v)$ .

The following theorem is a key step in the proof of Theorem 1.6.

**Theorem 5.1.** *Given any integer  $k \geq 3$  and  $0 < \eta \ll \epsilon_0$ , the following holds for all sufficiently large even integers  $n$ . Suppose  $H = (V_1 \cup \dots \cup V_k, E)$  is a  $k$ -partite  $k$ -graph with  $|V_1| = \dots = |V_k| = n$  and a vertex bipartition  $A \cup B$  such that*

- (i)  $|A_1| = |A_2| = n/2$ ,
- (ii)  $|A_i| = 0$  or  $\epsilon_0 n \leq |A_i| \leq (1 - \epsilon_0)n$  for  $i \geq 3$ ,
- (iii) for any even vector  $\mathbf{v}$ ,  $\bar{\delta}_1(H(\mathbf{v})) \leq \eta n^{k-1}$ .

*Then  $H$  contains a matching of size  $n - 1$ . Furthermore, if  $|A|$  is even, then  $H$  contains a perfect matching.*

To prove Theorem 5.1, we need the following simple result.

**Lemma 5.2.** *Given a set  $V$  of  $kn$  vertices for some event integer  $n$ , let  $V_1 \cup \dots \cup V_k$  and  $A \cup B$  be two partitions of  $V$  such that  $|V_1| = \dots = |V_k| = n$  and  $|A_1| = |A_2| = n/2$ , where  $A_i := A \cap V_i$  and  $B_i := B \cap V_i$ . Let  $H = (V, E_{\text{even}}(A, B))$ . If  $|A|$  is odd, then  $H$  contains a matching of size  $n - 1$ ; if  $|A|$  is even, then  $H$  contains a perfect matching.*

*Proof.* We arbitrarily partition  $V_3 \cup \dots \cup V_k$  into  $n$  crossing  $(k - 2)$ -sets and denote them by  $\mathcal{T}$ . Since  $|A_1| + |A_2| = n$  is even, it follows that  $\sum_{i \in [3, k]} |A_i| \equiv |A| \pmod{2}$ . First assume that  $|A|$  is even. Then  $\mathcal{T}$  contains an even number of odd members, and consequently, an even number of even members (because  $n$  is even). Thus we can group the members of  $\mathcal{T}$  into  $n/2$  pairs such that two sets in each pair have the same parity. We extend the two  $(k - 2)$ -sets in each pair to two even  $k$ -sets by adding four vertices, one from each of  $A_1, A_2, B_1, B_2$  – this is possible because the two  $(k - 2)$ -sets have the same parity. We choose different vertices for each pair so that the resulting  $n$   $k$ -sets are pairwise disjoint, – this gives the desired perfect matching of  $H$ .

Second assume that  $|A|$  is odd. Then  $\mathcal{T}$  consists of an odd number of odd sets, and consequently, an odd number of even sets. We can group the members of  $\mathcal{T}$  into  $n/2$  pairs such that in all but one pair, two members have the same parity. As in the case when  $\mathcal{T}$  is even, we extend the two  $(k - 2)$ -sets of  $\mathcal{T}$  with the same parity to two even  $k$ -sets by adding one vertex from each of  $A_1, A_2, B_1, B_2$ . Let  $T$  denote the even set in the last pair of  $\mathcal{T}$ . We form an even  $k$ -set with  $T$  and two remaining vertices in  $A_1 \cup A_2$ . This gives a matching of  $H$  of size  $n - 1$ .  $\square$

We also need the following result of Daykin and Häggkvist [4] in the proof of Theorem 5.1.

**Theorem 5.3.** [4] *For any integer  $k \geq 3$ , the following holds for sufficiently large integer  $n$ . If  $H$  is a  $k$ -partite  $k$ -graph with  $n$  vertices in each part such that  $\delta_1(H) \geq (1 - 1/k)(n^{k-1} - 1)$ , then  $H$  contains a perfect matching.*

*Proof of Theorem 5.1.* We first note that for any  $\mathbf{v} \in EV_k$  and any subset  $U_i \subseteq V_i^{\mathbf{v}}$ ,  $i \in [k]$ , such that  $|U_i| \geq \eta^{1/(2k)}n$ , by (iii), we have

$$(5.1) \quad \overline{\delta_1}(H[U_1, \dots, U_k]) \leq \eta n^{k-1} \leq \eta^{1/2} \prod_{2 \leq i \leq k} |U_i|.$$

We now apply Lemma 5.2 to  $H' := (V, E_{\text{even}}(A, B))$  and conclude that  $H'$  contains a matching  $M$  of size at least  $n - 1$ ; moreover,  $M$  is perfect if  $|A|$  is even. Let  $S := V \setminus V(M)$ . For each  $\mathbf{v} \in EV_k$ , let  $m_{\mathbf{v}}$  be the number of edges in  $M$  with location vector  $\mathbf{v}$ . Then  $\sum_{\mathbf{v} \in EV_k} m_{\mathbf{v}} = |M|$  as all the edges in  $M$  are even.

It suffices to build a matching of  $H$  that consists of  $m_{\mathbf{v}}$  edges with location vector  $\mathbf{v}$  for each  $\mathbf{v} \in EV_k$ . Let us partition  $EV_k$  into  $\mathcal{V}_1 \cup \mathcal{V}_2$  such that  $\mathcal{V}_1$  consists of all  $\mathbf{v}$  with  $m_{\mathbf{v}} < \eta^{1/(2k)}n$ . For each  $\mathbf{v} \in \mathcal{V}_1$ , we greedily find a matching of size  $m_{\mathbf{v}}$  in  $H_{\mathbf{v}}$ . To see why this is possible, note that in total at most  $2^{k-1}\eta^{1/(2k)}n \leq \epsilon_0^2 n$  edges of  $M$  have their location vectors in  $\mathcal{V}_1$ . Consequently the number of the crossing  $(k - 1)$ -sets in  $V_2^{\mathbf{v}} \cup \dots \cup V_k^{\mathbf{v}}$  that intersect these edges is at most

$$\epsilon_0^2 n \sum_{2 \leq i \leq k} \prod_{2 \leq j \leq k, j \neq i} |V_j^{\mathbf{v}}| \leq (k - 1)\epsilon_0 \prod_{2 \leq i \leq k} |V_i^{\mathbf{v}}|$$

because  $|V_i^{\mathbf{v}}| \geq \epsilon_0 n$  for  $2 \leq i \leq k$ . By (5.1), for any  $v \in V_1^{\mathbf{v}}$ , we have  $\deg_{H(\mathbf{v})}(v) \geq (1 - \eta^{1/2}) \prod_{2 \leq i \leq k} |V_i^{\mathbf{v}}| > (k - 1)\epsilon_0 \prod_{2 \leq i \leq k} |V_i^{\mathbf{v}}|$  – this guarantees the existence of the desired matchings for all  $\mathbf{v} \in \mathcal{V}_1$ .

Next we arbitrarily divide the remaining vertices of  $V \setminus S$  into balanced vertex partitions  $U_{\mathbf{v}} = U_1^{\mathbf{v}} \cup \dots \cup U_k^{\mathbf{v}}$ ,  $\mathbf{v} \in \mathcal{V}_2$ , such that  $U_i^{\mathbf{v}} \subseteq V_i^{\mathbf{v}}$  and  $|U_1^{\mathbf{v}}| = \dots = |U_k^{\mathbf{v}}| = m_{\mathbf{v}}$  – this is possible because  $\sum_{\mathbf{v} \in EV_k} m_{\mathbf{v}} = |M|$ . By (5.1), we know that  $\delta_1(H[U_{\mathbf{v}}]) \geq (1 - \eta^{1/2})m_{\mathbf{v}}^{k-1} \geq (1 - 1/k)m_{\mathbf{v}}^{k-1}$  as  $\eta$  is small enough. We thus apply Theorem 5.3 to each  $H[U_{\mathbf{v}}]$  and get a perfect matching of  $H[U_{\mathbf{v}}]$ . Putting all the matchings that we obtained together gives a matching of size  $|M|$  in  $H$ .  $\square$

Now we prove Theorem 1.6.

*Proof of Theorem 1.6.* Pick a new constant  $\epsilon_0$  such that  $\epsilon \ll \epsilon_0 \ll 1/k$ . As in the proof of Theorem 1.5, we may assume that (4.1) holds. Suppose  $H = (V_1 \cup \dots \cup V_k, E)$  is a  $k$ -partite  $k$ -graph with  $n$  vertices in each part with  $a_i := \delta_{[k] \setminus \{i\}}(H)$  for all  $i \in [k]$ . Moreover, suppose  $H$  has a vertex bipartition  $A \cup B = V_1 \cup \dots \cup V_k$  such that

- (†)  $(1/2 - \epsilon)n \leq a_1, a_2, |A_1|, |A_2| \leq (1/2 + \epsilon)n$ , and
- (‡)  $|E_{\text{even}}(A, B) \setminus E| \leq \epsilon n^k$ .

Note that we obtain (‡) after switching  $A_1$  and  $B_1$  if  $|E_{\text{odd}}(A, B) \setminus E| \leq \epsilon n^k$ . Furthermore, the above two properties remain valid if we switch an even number of  $A_i$  with  $B_i$ . Thus we may switch  $A_1, A_i$  with  $B_1, B_i$  whenever  $|A_i| > |B_i|$  for some  $i \geq 3$ . This results in  $|A_i| \leq |B_i|$  for all  $i \geq 3$  eventually. Moreover, by Fact 2.4 (i) and (†), we know that  $\delta_1(H) \geq (1/2 - \epsilon)n^{k-1}$ .

We now define *atypical* vertices. Let  $W$  be the set of vertices  $u \in V$  such that there exists an even  $\mathbf{v} \in EV_k$  such that  $u \in V^{\mathbf{v}}$  and  $\overline{\deg}_{H(\mathbf{v})}(u) > \sqrt{\epsilon}n^{k-1}/2$ . Let  $W_0 := W \cap (V_1 \cup V_2)$ . By (‡), we have

$$(5.2) \quad |W_0| \leq |W| \leq \frac{k\epsilon n^k}{\sqrt{\epsilon}n^{k-1}/2} \leq 2k\sqrt{\epsilon}n.$$

When forming a matching of  $H$ , we prefer using the edges that intersect both  $A_1, A_2$  or both  $B_1, B_2$  – we will call them *horizontal edges* (correspondingly, the edges that intersect both  $A_1, B_2$  or both  $A_2, B_1$  are *diagonal*). We distinguish the vertices of  $W_0$  that lie in few horizontal edges from the rest of  $W_0$ . For  $i = 1, 2$ , let  $W_{A_i}$  be the set of vertices of  $W_0 \cap A_i$  that lie in less than  $\epsilon_0 n^{k-1}$  horizontal edges; similarly let  $W_{B_i}$  be the set of vertices of  $W_0 \cap B_i$  that lie in less than  $\epsilon_0 n^{k-1}$  horizontal edges.

Define  $B_1^0 := (B_1 \setminus W_{B_1}) \cup W_{A_1}$  and  $B_2^0 := (B_2 \setminus W_{B_2}) \cup W_{A_2}$ . Let  $B_i^0 := B_i$  for  $3 \leq i \leq k$ , and let  $A_i^0 := V_i \setminus B_i^0$  for  $1 \leq i \leq k$ . Let  $A^0 := \bigcup_{i \in [k]} A_i^0$  and  $B^0 := V \setminus A^0$ . Finally, let

$$q := |B_2^0| - |B_1^0| = |B_2| - |B_1| + |W_{A_2}| + |W_{B_1}| - |W_{A_1}| - |W_{B_2}|.$$

By (†) and (5.2),  $|q| \leq 2\epsilon n + 2k\sqrt{\epsilon}n \leq 3k\sqrt{\epsilon}n$ . By relabelling  $V_1$  and  $V_2$  if necessary, we may assume that  $q \geq 0$ . Note that we still have  $|A_i| \leq |B_i|$  for all  $i \geq 3$ .

Our goal is to remove a small matching and possibly some crossing  $k$ -sets (non-edges) from  $H$  such that we can apply Theorem 5.1 to the remaining subgraph of  $H$ . To achieve the goal, we conduct the following five steps: we remove disjoint matchings  $M_1, \dots, M_4$  in the first four steps and a balanced vertex set  $S_5$  in the fifth step. For  $1 \leq j \leq 4$ , we define  $A^j := A^{j-1} \setminus V(M_j)$ ,  $B^j := B^{j-1} \setminus V(M_j)$ , and  $V^j := A^j \cup B^j$ . Let  $A^5 := A^4 \setminus S_5$ ,  $B^5 := B^4 \setminus S_5$  and  $V^5 := A^5 \cup B^5$ . For  $1 \leq j \leq 5$  and  $1 \leq i \leq k$ , define  $A_i^j := A^j \cap V_i$ ,  $B_i^j := B^j \cap V_i$ , and  $V_i^j := A_i^j \cup B_i^j$ .

*Step 1. Reducing the gap between  $|B_1^0|$  and  $|B_2^0|$ .* Our first matching  $M_1$  is crucial on balancing the sizes of  $B_1^0$  and  $B_2^0$ , and this is the only place that we need the exact codegree condition. Let  $H_1 := H[A_1^0 \cup B_2^0 \cup V_3 \cup \dots \cup V_k]$  and note that

$$\delta_{[k] \setminus \{1\}}(H_1) \geq a_1 - |B_1^0|, \delta_{[k] \setminus \{2\}}(H_1) \geq a_2 - (n - |B_2^0|)$$

and  $\delta_{[k] \setminus \{i\}}(H_1) = a_i$  for  $3 \leq i \leq k$ . So we have

$$\sum_{i=1}^k \delta_{[k] \setminus \{i\}}(H_1) \geq \sum_{i=1}^k a_i + |B_2^0| - |B_1^0| - n = q - n + \sum_{i=1}^k a_i.$$

We have

$$q - n + \sum_{i=1}^k a_i \leq q \leq 3k\sqrt{\epsilon}n < \min_{i \in [k]} |V(H_1) \cap V_i| - k.$$

If  $q - n + \sum_{i=1}^k a_i > 0$ , then Fact 1.3 provides a matching  $M'$  of size  $q - n + \sum_{i=1}^k a_i$  in  $H_1$ . Let  $M_1 := M'$  if  $\sum_{i=1}^k a_i \leq n$  and let  $M_1 \subseteq M'$  be a (sub)matching of size  $q$  if  $\sum_{i=1}^k a_i > n$ . Otherwise let  $M_1 := \emptyset$ . So we have  $|M_1| \leq q \leq 3k\sqrt{\epsilon}n$  in all cases.

*Step 2. Cleaning  $V_1$  and  $V_2$ .* In this step we find a matching  $M_2$  that covers all the remaining vertices of  $W_0$  and uses the same amount of the vertices from  $A_1^0$  and  $A_2^0$ . Let  $W'_0 := (W_{A_1} \cup W_{A_2} \cup W_{B_1} \cup W_{B_2}) \setminus V(M_1)$  and  $W''_0 := W_0 \setminus (W'_0 \cup V(M_1))$ . We cover the vertices of  $W''_0$  and  $W'_0$  as follows.

- (1) By definition, each vertex  $u \in W''_0$  lies in at least  $\epsilon_0 n^{k-1}$  horizontal edges, i.e., those that intersect both  $A_1$  and  $A_2$ , or intersect both  $B_1$  and  $B_2$ . By (5.2), among these edges, at least  $\epsilon_0 n^{k-1}/2$  horizontal edges do not intersect  $W$ , so they intersect both  $A_1^0$  and  $A_2^0$ , or intersect both  $B_1^0$  and  $B_2^0$ . We greedily cover the vertices of  $W''_0$  by these disjoint horizontal edges.

- (2) Since  $\delta_1(H) \geq (1/2 - \epsilon)n^{k-1}$ , by definition, every vertex  $u \in W'_0$  lies in at least  $(1/2 - \epsilon)n^{k-1} - \epsilon_0 n^{k-1}$  diagonal edges. By the definitions of  $A_1^0, A_2^0, B_1^0, B_2^0$ , each  $u \in W'_0$  lies in at least  $(1/2 - \epsilon)n^{k-1} - \epsilon_0 n^{k-1} - |W_0|n^{k-2} \geq \epsilon_0 n^{k-1}$  edges that intersect both  $A_1^0$  and  $A_2^0$ , or both  $B_1^0$  and  $B_2^0$ . We greedily cover the vertices of  $W'_0$  by such edges.

To see why the above process is possible, we note that when finding an edge for a vertex  $u$ , the number of vertices that we need to avoid is at most  $|V(M_1)| + k|W_0| \leq 3k^2\sqrt{\epsilon}n + 2k^2\sqrt{\epsilon}n \leq 5k^2\sqrt{\epsilon}n$  by (5.2) and  $|M_1| \leq 3k\sqrt{\epsilon}n$ . Hence these vertices lie in at most  $5k^2\sqrt{\epsilon}n^{k-1} < \epsilon_0 n^{k-1}/2$  crossing  $(k-1)$ -sets, so we can find an edge that covers  $u$  and avoids all the existing edges.

Let us bound  $|B_2^2| - |B_1^2| = |B_2^0 \setminus V(M_1 \cup M_2)| - |B_1^0 \setminus V(M_1 \cup M_2)|$ . By the definition of  $M_1$ , we have  $|B_1^0 \cap V(M_1)| = 0$  and  $|B_2^0 \cap V(M_1)| = |M_1|$ . By the definition of  $M_2$ , we have  $|B_1^0 \cap V(M_2)| = |B_2^0 \cap V(M_2)|$ . Thus,

$$|B_2^2| - |B_1^2| = |B_2^0| - |B_1^0| - |M_1| = q - |M_1|.$$

Note that

$$q - |M_1| = \begin{cases} n - \sum_{i=1}^k a_i & \text{if } q - n + \sum_{i=1}^k a_i \geq 0 \text{ and } \sum_{i=1}^k a_i \leq n; \\ 0 & \text{if } q - n + \sum_{i=1}^k a_i \geq 0 \text{ and } \sum_{i=1}^k a_i > n; \\ q \leq n - \sum_{i=1}^k a_i & \text{if } q - n + \sum_{i=1}^k a_i < 0. \end{cases}$$

So we have

$$(5.3) \quad 0 \leq |B_2^2| - |B_1^2| = q - |M_1| \leq \max \left\{ 0, n - \sum_{i=1}^k a_i \right\}.$$

*Step 3. Cleaning  $V_3, \dots, V_k$ .* Let  $X$  consist of all  $x \in A_i \setminus W$  for some  $i \geq 3$  such that  $|A_i| \leq 2\epsilon_0 n$ . In this step we build a matching  $M_3$  that covers all the remaining vertices of  $W$  and the vertices of  $X$ .

Let  $W'$  be the set of vertices in  $(W_3 \cup \dots \cup W_k) \setminus V(M_1 \cup M_2)$  that are contained in at least  $3k^2\epsilon_0 n^{k-1}$  horizontal edges. Let  $W'' := (W_3 \cup \dots \cup W_k) \setminus (V(M_1 \cup M_2) \cup W')$ . Since  $\delta_1(H) \geq (1/2 - \epsilon)n^{k-1}$ , by definition, each  $u \in W''$  is contained in at least  $(1/2 - \epsilon)n^{k-1} - 3k^2\epsilon_0 n^{k-1}$  diagonal edges. Note that by  $(\dagger)$ , we have  $|A_1||B_2| \leq (1/4 + 3\epsilon)n^2$  and  $|B_1||A_2| \leq (1/4 + 3\epsilon)n^2$ . Then since  $u$  lies in at most  $|A_1||B_2|n^{k-3} \leq (1/4 + 3\epsilon)n^{k-1}$  edges that intersect both  $A_1$  and  $B_2$  and at most  $|B_1||A_2|n^{k-3} \leq (1/4 + 3\epsilon)n^{k-1}$  edges that intersect both  $B_1$  and  $A_2$ , there are at least  $3k^2\epsilon_0 n^{k-1}$  edges that contain  $u$  and intersect both  $A_1$  and  $B_2$ , and at least  $3k^2\epsilon_0 n^{k-1}$  edges that contain  $u$  and intersect both  $B_1$  and  $A_2$ . Finally, for any vertex  $x \in X \setminus W$ , assume that  $x \in A_i$  for some  $3 \leq i \leq k$ . Since the binary vectors  $\mathbf{v} \in \{0, 1\}^k$  with exactly two 1's are even, the fact that  $x \notin W$  implies that  $x$  is contained in at least

$$\prod_{j \in [k] \setminus \{1, i\}} |B_j| \cdot |A_1| - \frac{1}{2}\sqrt{\epsilon}n^{k-1} \geq \frac{n^{k-1}}{2^k} - \frac{1}{2}\sqrt{\epsilon}n^{k-1} > 3k^2\epsilon_0 n^{k-1}$$

edges in  $A_1 \cup (\bigcup_{2 \leq j \leq k, j \neq i} B_j) \cup A_i$ , and in at least  $3k^2\epsilon_0 n^{k-1}$  edges in  $B_1 \cup A_2 \cup (\bigcup_{3 \leq j \leq k, j \neq i} B_j) \cup A_i$ , where we used  $|A_1||B_2| \geq n^2/8$  and  $|B_i| \geq n/2$  for  $3 \leq i \leq k$  in the first inequality.

- (1) We first greedily find  $|W'|$  disjoint horizontal edges such that each of them contains one vertex of  $W'$  but no other vertices from  $W \cup X \cup V(M_1 \cup M_2)$ .
- (2) Next, we split  $W'' \cup X$  arbitrarily to  $W''_1$  and  $W''_2$  of sizes  $\lfloor |W'' \cup X|/2 \rfloor$  and  $\lceil |W'' \cup X|/2 \rceil$ , respectively. We greedily find  $|W''_1|$  disjoint edges such that each of them contains one vertex  $u \in W''_1$ , one vertex from each of  $B_1$  and  $A_2$ , but no other vertices from  $W \cup X \cup V(M_1 \cup M_2)$ ; in particular, if  $u \in A_i \subseteq X$ , then the edge is taken in  $B_1 \cup A_2 \cup (\bigcup_{3 \leq j \leq k, j \neq i} B_j) \cup A_i$ .

Finally we greedily find  $|W''_2|$  disjoint edges such that each of them contains one vertex  $u \in W''_2$ , one vertex from each of  $A_1$  and  $B_2$ , but no other vertices from  $W \cup X \cup V(M_1 \cup M_2)$ ; in particular, if  $u \in A_i \subseteq X$ , then the edge is taken in  $A_1 \cup (\bigcup_{2 \leq j \leq k, j \neq i} B_j) \cup A_i$ .

The above process is possible because when considering a vertex  $u$ , the number of vertices that we need to avoid is at most  $|V(M_1)| + k|W| + k \cdot 2(k-2)\epsilon_0 n < 3k^2\epsilon_0 n$  because of (5.2), the facts  $|M_1| \leq 3k\sqrt{\epsilon}n$  and  $|X| \leq 2(k-2)\epsilon_0 n$ . Hence these vertices lie in less than  $3k^2\epsilon_0 n^{k-1}$   $(k-1)$ -sets, so we can always find a desired edge that covers  $u$  and avoids all the existing edges.

*Step 4. Balancing the sizes of  $B_1^3$  and  $A_2^3$ .* Let  $m := |B_1^3| - |A_2^3|$ . We find a matching  $M_4$  of size  $|m|$  as follows. If  $m \geq 0$ , then  $M_4$  consists of  $m$  disjoint edges from  $B^3$  that are disjoint from  $M_1 \cup M_2 \cup M_3$ .

Since  $(0, \dots, 0) \in EV_k$ , this can be done since  $H[(0, \dots, 0)]$  is almost complete. Otherwise  $M_4$  consists of  $|m|$  disjoint edges with location vector  $(1, 1, 0, \dots, 0)$  that are disjoint from  $M_1 \cup M_2 \cup M_3$  – this is possible because  $H[(1, 1, 0, \dots, 0)]$  is almost complete.

After removing  $M_4$ , the resulting sets  $B_1^4$  and  $A_2^4$  satisfy  $|B_1^4| = |A_2^4|$ . Now let us bound  $|B_2^4| - |B_1^4|$ . The definition of  $M_3$  implies that

$$-1 \leq |B_1^2 \cap V(M_3)| - |B_2^2 \cap V(M_3)| \leq 0;$$

the definition of  $M_4$  implies that  $|B_1^2 \cap V(M_4)| = |B_2^2 \cap V(M_4)|$ . By (5.3),

$$\begin{aligned} -1 \leq |B_2^4| - |B_1^4| &= |B_2^2| - |B_1^2| + |B_1^2 \cap V(M_3)| - |B_2^2 \cap V(M_3)| \\ &\leq \max \left\{ 0, n - \sum_{i=1}^k a_i \right\} \leq k - 3. \end{aligned}$$

*Step 5. Balancing the sizes of  $B_1^4$  and  $B_2^4$ .* Let  $t := |B_2^4| - |B_1^4|$  and thus  $-1 \leq t \leq k - 3$ . If  $t > 0$ , then  $n - \sum_{i=1}^k a_i \geq t > 0$ . Let  $S_5$  be a  $kt$ -set in  $V^4$  with  $t$  vertices from each of  $A_1^4, B_2^4$ , and  $V_i^4$  for  $3 \leq i \leq k$  such that  $|A^4 \setminus S_5|$  is even. The requirement that  $|A^4 \setminus S_5|$  is even can be easily fulfilled if any  $A_i^4$ ,  $i \geq 3$ , is not empty. On the other hand, if all  $A_3^4, \dots, A_k^4$  are empty, then since  $|B_2^4 \setminus S_5| = |B_1^4|$ , we have  $|A_1^4 \setminus S_5| = |A_2^4|$  and consequently,

$$|A^4 \setminus S_5| = |A_1^4 \setminus S_5| + |A_2^4| = 2|A_2^4|$$

is even. Since  $t \leq n - \sum_{i=1}^k a_i$ , to complete the proof, it suffices to find a perfect matching in  $V^5 := V^4 \setminus S_5$ . If  $t = -1$ , then let  $S_5$  be a  $k$ -set with one vertex from each of  $B_1^4, A_2^4$  and  $V_i^4$  for  $3 \leq i \leq k$  such that  $|A^4 \setminus S_5|$  is even – this can be achieved by the same argument as in the  $t > 0$  case. Again it suffices to find a perfect matching in  $V^5$ . At last, if  $t = 0$  then set  $S_5 = \emptyset$ . In this case  $|A^5| = |A^4 \setminus S_5|$  may be odd; however, it suffices to find a matching of size  $|V^5| - 1$  in  $H[V^5]$ . In summary, it remains to find a perfect matching in  $H[V^5]$  if  $|A^5|$  is even and a matching of size  $|V^5| - 1$  otherwise. This follows from Theorem 5.1 immediately and thus it remains to verify the assumptions of Theorem 5.1.

Let  $n' := |V_1^5|$  and  $H' := H[V^5]$ . Note that

$$|M_1 \cup M_2 \cup M_3| \leq |M_1| + |W| + |X| \leq 3k\sqrt{\epsilon}n + 2k\sqrt{\epsilon}n + 2k\epsilon_0n, \quad \text{and}$$

$$|M_4| = ||B_1^3| - |A_2^3|| \leq |B_1| - |A_2| + |M_1 \cup M_2 \cup M_3| \leq 3k\epsilon_0n,$$

where  $|B_1| - |A_2| \leq 2\epsilon n$  by  $(\dagger)$ . Note that we have  $V(M_4) \cap A_i = \emptyset$  for  $3 \leq i \leq k$ , and when building  $M_3$ , we may use the vertices of  $A_i$ ,  $3 \leq i \leq k$ , of size at least  $2\epsilon_0n$  only when we cover the vertices of  $W$ . Thus for  $3 \leq i \leq k$ , if  $|A_i^5| \neq 0$ , then

$$\begin{aligned} |A_i^5| &\geq |A_i| - |M_1| - |V(M_2 \cup M_3) \cap A_i| - |S_5 \cap A_i| \\ &\geq 2\epsilon_0n - 3k\sqrt{\epsilon}n - 2k\sqrt{\epsilon}n - (k - 3) \geq \epsilon_0n \geq \epsilon_0n', \end{aligned}$$

and  $|A_i^5| \leq |A_i| \leq n/2 < (1 - \epsilon_0)n'$ . Moreover, by the choice of  $M_4$  and  $S_5$ , we have  $|A_2^5| = |B_1^5| = |B_2^5|$ , and thus  $|A_1^5| = |A_2^5| = n'/2$ . In particular, this means that  $n'$  is even. Finally, note that

$$\begin{aligned} n' &= n - |M_1| - |M_2| - |M_3| - |M_4| - |S_5|/k \\ &\geq n - 3k\sqrt{\epsilon}n - 2k\sqrt{\epsilon}n - 2k\epsilon_0n - 3k\epsilon_0n - (k - 3) \geq (1 - 6k\epsilon_0)n. \end{aligned}$$

So for any  $\mathbf{v} \in EV_k$ , and any vertex  $u \in H'(\mathbf{v})$ , since  $u \notin W$  and  $n$  is large enough, we have

$$\deg_{H'(\mathbf{v})}(u) \leq \sqrt{\epsilon}n^{k-1}/2 < \sqrt{\epsilon}n'^{k-1}.$$

So we are done by Theorem 5.1 with  $\eta = \sqrt{\epsilon}$ .  $\square$

## REFERENCES

- [1] R. Aharoni, A. Georgakopoulos, and P. Sprüssel. Perfect matchings in  $r$ -partite  $r$ -graphs. *European J. Combin.*, 30(1):39–42, 2009.
- [2] N. Alon, P. Frankl, H. Huang, V. Rödl, A. Ruciński, and B. Sudakov. Large matchings in uniform hypergraphs and the conjecture of Erdős and Samuels. *J. Combin. Theory Ser. A*, 119(6):1200–1215, 2012.
- [3] A. Czygrinow and V. Kamat. Tight co-degree condition for perfect matchings in 4-graphs. *Electron. J. Combin.*, 19(2):Paper 20, 16, 2012.
- [4] D. E. Daykin and R. Häggkvist. Degrees giving independent edges in a hypergraph. *Bull. Austral. Math. Soc.*, 23(1):103–109, 1981.



- [5] H. Hàn, Y. Person, and M. Schacht. On perfect matchings in uniform hypergraphs with large minimum vertex degree. *SIAM J. Discrete Math.*, 23:732–748, 2009.
- [6] J. Han. Decision problem for perfect matchings in dense uniform hypergraphs. *Trans. Amer. Math. Soc.*, *accepted*.
- [7] J. Han. Near perfect matchings in  $k$ -uniform hypergraphs. *Combin. Probab. Comput.*, 24(5):723–732, 2015.
- [8] J. Han. Near perfect matchings in  $k$ -uniform hypergraphs II. *SIAM J. Discrete Math.*, 30:1453–1469, 2016.
- [9] J. Han. Perfect matchings in hypergraphs and the erdős matching conjecture. *SIAM J. Discrete Math.*, 30:1351–1357, 2016.
- [10] J. Han and A. Treglown. The complexity of perfect matchings and packings in dense hypergraphs. *arXiv:1609.06147*.
- [11] P. Keevash and R. Mycroft. A geometric theory for hypergraph matching. *Mem. Amer. Math. Soc.*, 233(1098):vi+95, 2015.
- [12] I. Khan. Perfect matchings in 3-uniform hypergraphs with large vertex degree. *SIAM J. Discrete Math.*, 27(2):1021–1039, 2013.
- [13] I. Khan. Perfect matchings in 4-uniform hypergraphs. *J. Combin. Theory Ser. B*, 116:333–366, 2016.
- [14] D. Kühn and D. Osthus. Matchings in hypergraphs of large minimum degree. *J. Graph Theory*, 51(4):269–280, 2006.
- [15] D. Kühn, D. Osthus, and A. Treglown. Matchings in 3-uniform hypergraphs. *J. Combin. Theory Ser. B*, 103(2):291–305, 2013.
- [16] A. Lo and K. Markström. F-factors in hypergraphs via absorption. *Graphs Combin.*, 31(3):679–712, 2015.
- [17] H. Lu, Y. Wang, and X. Yu. Almost perfect matchings in  $k$ -partite  $k$ -graphs. *arXiv:1608.04838*.
- [18] K. Markström and A. Ruciński. Perfect Matchings (and Hamilton Cycles) in Hypergraphs with Large Degrees. *European J. Comb.*, 32(5):677–687, July 2011.
- [19] O. Pikhurko. Perfect matchings and  $K_4^3$ -tilings in hypergraphs of large codegree. *Graphs Combin.*, 24(4):391–404, 2008.
- [20] V. Rödl and A. Ruciński. Dirac-type questions for hypergraphs a survey (or more problems for endre to solve). *An Irregular Mind*, Bolyai Soc. Math. Studies 21:561–590, 2010.
- [21] V. Rödl, A. Ruciński, and E. Szemerédi. A Dirac-type theorem for 3-uniform hypergraphs. *Combin. Probab. Comput.*, 15(1-2):229–251, 2006.
- [22] V. Rödl, A. Ruciński, and E. Szemerédi. Perfect matchings in uniform hypergraphs with large minimum degree. *European J. Combin.*, 27(8):1333–1349, 2006.
- [23] V. Rödl, A. Ruciński, and E. Szemerédi. Perfect matchings in large uniform hypergraphs with large minimum collective degree. *J. Combin. Theory Ser. A*, 116(3):613–636, 2009.
- [24] A. Treglown and Y. Zhao. Exact minimum degree thresholds for perfect matchings in uniform hypergraphs. *J. Combin. Theory Ser. A*, 119(7):1500–1522, 2012.
- [25] A. Treglown and Y. Zhao. Exact minimum degree thresholds for perfect matchings in uniform hypergraphs II. *J. Combin. Theory Ser. A*, 120(7):1463–1482, 2013.
- [26] A. Treglown and Y. Zhao. A note on perfect matchings in uniform hypergraphs. *Electron. J. Combin.*, 23:P1.16, 2016.
- [27] C. Zang. *Matchings and Tilings in hypergraphs*. PhD thesis, Georgia State University, 2016.
- [28] Y. Zhao. Recent advances on dirac-type problems for hypergraphs. In *Recent Trends in Combinatorics*, volume 159 of the *IMA Volumes in Mathematics and its Applications*. Springer, New York, 2016.

INSTITUTO DE MATEMÁTICA E ESTATÍSTICA, UNIVERSIDADE DE SÃO PAULO, RUA DO MATÃO 1010, 05508-090, SÃO PAULO, BRAZIL

*E-mail address*, Jie Han: `jhan@ime.usp.br`

DEPARTMENT OF MATHEMATICS AND STATISTICS, GEORGIA STATE UNIVERSITY, ATLANTA, GA 30303

*E-mail address*, Chuanyun Zang: `czang1@student.gsu.edu`

*E-mail address*, Yi Zhao: `yzhao6@gsu.edu`